

Deriving CSW rules for massive scalar legs and pure Yang-Mills loops

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ABSTRACT: This article provides two complementary detailed derivations of Cachazo-Svrček-Witten-style Feynman rules for Yang-Mills gauge theory coupled to a massive coloured scalar as presented in earlier work. These proceed through a direct canonical transformation method on space-time and through a gauge transformation in an action constructed on twistor space. It is shown explicitly that the field transformations are identical in both cases. Some simple tree-level examples of our rules are given and we comment on the application of them to the calculation of the rational part of one-loop pure glue amplitudes. A possible direct quantum completion of pure glue CSW rules based on dimensional regularisation motivated by these results is sketched. Finally, it is shown how to derive CSW rules for effective Higgs-gluon and Higgs-matter couplings proposed in the literature directly from the action. This derivation yields additional towers of vertices which generate a subset of the contributions to effective multi-Higgs scattering amplitudes.

KEYWORDS: Gauge Symmetry, QCD, NLO Computations, Higgs Physics.

Contents

1. Introduction	2
2. Lagrangian based derivations of CSW rules	3
2.1 Spinor, light cone and twistor conventions	3
2.2 CSW rules from a canonical transformation	7
2.3 CSW rules from twistor Yang-Mills theory	10
2.4 Comparison of the two approaches	12
2.5 Equivalence theorem violations	16
3. CSW rules for massive scalars	18
3.1 Summary of the rules	18
3.2 Tree level examples	19
3.3 Light-cone derivation of CSW rules for massive scalars	21
3.4 Twistor-Yang Mills with massive scalars	26
3.5 Equivalence theorem for massive scalars	29
4. One-loop amplitudes in pure Yang-Mills	30
4.1 The four-point all-plus amplitude in pure Yang-Mills	32
4.2 Remarks on other amplitudes	35
4.3 Direct construction of pure Yang-Mills amplitudes	36
5. Higgs-gluon couplings	39
5.1 Twistor Yang-Mills derivation	40
5.2 Extension to more general vertices	42
5.3 Coupling in other matters	43
5.4 Charged scalar vertices	44
5.5 Comment on canonical transformation method	45
6. Conclusions	45
A. Yang-Mills conventions	46
B. Spinor identities	47
C. Derivation of the transformation of $A_{\bar{z}}$ in the canonical method	47
D. Harmonic expansions and Noether procedure for actions on twistor space	49
D.1 Coupling the scalar twistor fields into Yang-Mills	52
D.2 Lifting formula	53

1. Introduction

Scattering amplitudes in Yang-Mills theories can take much simpler forms than suggested by individual textbook Feynman diagrams. A prominent example are the maximally helicity violating (MHV) tree-level amplitudes with two negative helicity gluons and an arbitrary number of positive helicity gluons that are given by a simple closed-form expression [1, 2]. Although Nair was able to interpret this simple formula in terms of a $\mathcal{N} = 4$ supersymmetric two-dimensional sigma-model [3] on a degree one curve embedded in twistor space, until recently no systematic way was known to exploit these results for calculations of general scattering amplitudes. This situation improved when Witten generalised Nair’s observation and obtained a precise relation between all tree level Yang-Mills scattering amplitudes on space-time and, in general, higher degree curves in twistor space [4]. Since twistor space is the space on which the conformal group acts linearly, this reflects the underlying conformal invariance of Yang-Mills theory at tree level.

Witten’s twistor observations and his speculations about a possible underlying twistor string theory led to the development of new Feynman-like rules by Cachazo, Svrček and Witten (CSW) [5] where the MHV amplitudes serve as vertices. The tree-level CSW rules were applied also to amplitudes with massless quarks and scalars in supersymmetric theories [6–8], currents with external W and Z vector bosons or the Higgs boson [9–11] and found numerous further applications [12–15]. At the one-loop level, a construction of the cut-constructable pieces of amplitudes from MHV vertices was possible [16–22] while the CSW representation of the rational piece of amplitudes was initially unknown. In pure Yang-Mills theory, the tree-level CSW formalism was shown to follow from the on-shell recursion relations of Britto, Cachazo, Feng and Witten (BCFW) [23, 24] both indirectly [24] and directly [25]. However, subsequently it was understood how to derive the CSW rules directly from a Lagrangian, independent of the BCFW rules [26–28]. These action based approaches led to several proposals for the construction of the rational parts of one loop amplitudes [29–32] and were furthermore applied to supersymmetric theories [33, 34] and to Einstein gravity [35] or self-dual $\mathcal{N} = 8$ supergravity [36].

In a previous paper [37] we have presented CSW-style rules for amplitudes with colored massive scalars, incorporating for the first time propagating massive particles in this approach. The rules have been obtained using the two different Lagrangian approaches in the literature [27, 28], demonstrating that these methods are useful to derive new results. Apart from providing a simple testing ground for ideas, amplitudes with massive scalars are intrinsically interesting since they are related by supersymmetry to phenomenologically relevant tree amplitudes with massive quarks [38] and to the rational part of pure Yang-Mills amplitudes at one loop [39]. The scalar mass term in the Lagrangian was shown to give rise to a new tower of vertices with an arbitrary number of positive helicity gluons, in addition to the MHV vertices present already for massless scalars. Like the MHV vertices in massless theories, the new vertices resulting from the mass term only contain “holomorphic” spinor products and therefore localise on lines in twistor space, whereas the structure of the simplest on-shell scattering amplitudes of massive scalars is more involved [40–42]. Therefore, in contrast to the massless case, it appears difficult to derive the rules given

in [37] from the BCFW relations along the lines of [25]. On the contrary, the holomorphic representation allows to give a far more direct proof of the BCFW recursion for amplitudes with massive scalars [37] compared to earlier treatments [40, 43].

In this paper we provide full details of the derivation of the massive CSW rules through the two different action based methods available in the literature. In one approach [27] a canonical transformation is used to bring the Yang-Mills Lagrangian in light-cone gauge to a form which appears to involve only MHV vertices, as has been explicitly verified for the first five vertices [44]. The second approach [28] starts from an action written directly on the twistor space [45] that reduces to the space-time Yang-Mills action in a particular gauge [33, 28] while a different gauge choice results in the CSW rules. The two approaches have been known to be closely related [30] and in this article we will show equivalence explicitly by deriving the canonical transformation coefficients of [44] from the twistor lifting formulae.

We also study the applicability of our rules to the calculation of the rational terms of one-loop amplitudes in pure Yang-Mills theory by demonstrating explicitly that the four-point amplitude with positive helicity gluons is correctly reproduced, without the need to take “equivalence theorem violations” [32] into account or to abandon working with dimensional regularisation [31]. We argue that this feature persists for general amplitudes. As a further example of the derivation of CSW-rules we derive the rules of [9, 10] resulting from an effective Higgs-gluon coupling and obtain additional multi-Higgs-gluon CSW vertices not noted in the previous literature.

The rest of this paper is organised as follows: We begin in section 2 by setting up our conventions and notation, proceeding to review the canonical transformation method and the construction of a twistor action to derive the CSW rules. This will culminate in a direct comparison between the field transformations which are shown to coincide exactly. The section is closed by a simple observation about violations of the equivalence theorem. In the next section we derive the CSW rules for a massive colored scalar [37] in the same two ways as demonstrated earlier for glue. Results obtained here are then applied in section 4 to calculate one-loop scattering amplitudes in Yang-Mills theory. A final section on effective Higgs-gluon couplings, section 5, leaves nothing but the conclusions and some technical appendices.

2. Lagrangian based derivations of CSW rules

2.1 Spinor, light cone and twistor conventions

We begin by setting up our general conventions for two-component spinors, the light-cone decomposition used for the space-time derivation of the CSW Lagrangian and the conventions used for the twistor Yang-Mills approach.

To every four-momentum we can associate a matrix $p^{\alpha\dot{\beta}} = p_\mu \bar{\sigma}^{\mu\alpha\dot{\beta}}$. For a light-like momentum the matrix $p^{\alpha\dot{\alpha}}$ factorises into a product of two spinors π_p^α and $\pi_p^{\dot{\alpha}}$ that are determined up to a rescaling $\pi^{\dot{\alpha}} \rightarrow \lambda \pi^{\dot{\alpha}}$ and $\pi^\alpha \rightarrow \lambda^{-1} \pi^\alpha$ with $\lambda \in \mathbb{C}$ that leaves the momentum $p^{\dot{\alpha}\beta}$ invariant. Spinors with lower indices are defined by $\pi_{\dot{\beta}} = \pi^{\dot{\alpha}} \varepsilon_{\dot{\alpha}\dot{\beta}}$ and $\pi_\beta = \pi^\alpha \varepsilon_{\alpha\beta}$. We also use the bra-ket notation $\pi_{p\dot{\alpha}} = |p+\rangle$, $\pi_p^\alpha = |p-\rangle$, $\pi_{p\alpha} = \langle p+|$

and $\pi_p^{\dot{\alpha}} = \langle p- |$ familiar from the QCD literature. We will refer to the dotted spinors as holomorphic spinors and the un-dotted as anti-holomorphic ones. These conventions follow [28] where the role of dotted and un-dotted indices is reversed compared to [4]. Spinor products are denoted as

$$\langle pq \rangle = \langle p- | q+ \rangle = \pi_p^{\dot{\alpha}} \pi_{q\dot{\alpha}}, \quad [qp] = \langle q+ | p- \rangle = \pi_{q\dot{\alpha}} \pi_p^{\dot{\alpha}}. \quad (2.1)$$

Given a basis $(\eta^{\dot{\alpha}}, \kappa^{\dot{\alpha}})$ of the holomorphic spinors and an anti-holomorphic basis $(\eta^{\alpha}, \kappa^{\alpha})$ normalised according to $\langle \eta \kappa \rangle = [\kappa \eta] = \sqrt{2}$ one can expand any momentum in terms of light cone components $(p_+, p_-, p_z, p_{\bar{z}})$ according to

$$p^{\alpha\dot{\alpha}} = p_- \eta^{\alpha} \eta^{\dot{\alpha}} + p_+ \kappa^{\alpha} \kappa^{\dot{\alpha}} + p_z \kappa^{\alpha} \eta^{\dot{\alpha}} + p_{\bar{z}} \eta^{\alpha} \kappa^{\dot{\alpha}} \quad (2.2)$$

The components of the momenta can be projected out by

$$2p_- = \kappa_{\alpha} p^{\alpha\dot{\alpha}} \kappa_{\dot{\alpha}}, \quad 2p_+ = \eta_{\alpha} p^{\alpha\dot{\alpha}} \eta_{\dot{\alpha}}, \quad 2p_z = -\eta_{\alpha} p^{\alpha\dot{\alpha}} \kappa_{\dot{\alpha}}, \quad 2p_{\bar{z}} = -\kappa_{\alpha} p^{\alpha\dot{\alpha}} \eta_{\dot{\alpha}} \quad (2.3)$$

With this conventions the Minkowski product is expressed in light-cone components as

$$p_{\mu} k^{\mu} = \frac{1}{2} p^{\alpha\dot{\alpha}} k_{\dot{\alpha}\alpha} = p_+ k_- + p_+ k_- - p_z k_{\bar{z}} - p_{\bar{z}} k_z \equiv p_+ k^+ + p_- k^- + p_z k^z + p_{\bar{z}} k^{\bar{z}} \quad (2.4)$$

The two-component spinors can be expanded in the (η, κ) bases as

$$\pi_p^{\alpha} = p_+^{-1/2} (p_{\bar{z}} \eta^{\alpha} + p_+ \kappa^{\alpha}), \quad \pi_p^{\dot{\alpha}} = p_+^{-1/2} (p_z \eta^{\dot{\alpha}} + p_+ \kappa^{\dot{\alpha}}) \quad (2.5)$$

up to an arbitrary phase. For negative or complex p_+ the square root should be interpreted as $p_+^{1/2} = |p_+|^{1/2} e^{i\phi_p/2}$ with the phase defined by $p_+ = |p_+| e^{i\phi_p}$. The expressions of spinor-products in terms of the light-cone components are given by

$$\begin{aligned} \langle pk \rangle &= \sqrt{2} p_+^{-1/2} k_+^{-1/2} (p_z k_+ - k_z p_+) \equiv \sqrt{2} p_+^{-1/2} k_+^{-1/2} (p, k) \\ [kp] &= \sqrt{2} p_+^{-1/2} k_+^{-1/2} (k_{\bar{z}} p_+ - p_{\bar{z}} k_+) \equiv \sqrt{2} p_+^{-1/2} k_+^{-1/2} \{k, p\} \end{aligned} \quad (2.6)$$

Here we also have introduced momentum brackets (p, k) and $\{p, k\}$ as used in [44]. We also have the relations

$$\langle \eta k \rangle = \sqrt{2 k_+} = [k \eta] \quad (2.7)$$

for Minkowski momenta.

In the light-cone formalism one can also implement an off-shell continuation of spinor products [27]. Since the momentum component $p_- = (p^2 + 2p_z p_{\bar{z}})/(2p_+)$ does not appear in the explicit representation of the spinors given in (2.5), these expressions are also defined for off-shell momenta. This corresponds to an off-shell continuation of the spinors defined by a light-cone projection of the momentum p according to

$$\pi_p^{\dot{\alpha}} \pi_p^{\alpha} = p^{\dot{\alpha}\alpha} - \frac{p^2}{2p_+} \eta^{\dot{\alpha}} \eta^{\alpha} \quad (2.8)$$

This prescription is equivalent [13] to the original off-shell continuation in the CSW rules [5].

For polarisation vectors satisfying the gauge condition $\eta \cdot \epsilon^\pm = 0$ we use the usual expressions of the spinor-helicity formalism

$$\epsilon^{+,\alpha\dot{\alpha}}(k) = \sqrt{2} \frac{\pi_k^\alpha \eta^{\dot{\alpha}}}{\langle \eta k \rangle} ; \quad \epsilon^{-,\dot{\alpha}\alpha}(k) = \sqrt{2} \frac{\eta^\alpha \pi_k^{\dot{\alpha}}}{[k\eta]} \quad (2.9)$$

The reference spinors η^α and $\eta^{\dot{\alpha}}$ are chosen to be identical for all gluons and are taken to be the same as in the light-cone decomposition (2.2). This has been called the 'space-cone' gauge in [46]. The relevant light-cone components of the polarisation vectors are trivial in our spinor conventions

$$(\epsilon^+(k))_z = -\frac{[\eta k]}{\langle \eta k \rangle} = 1 \quad (\epsilon^-(k))_{\bar{z}} = -\frac{\langle k\eta \rangle}{[k\eta]} = 1 \quad (2.10)$$

Therefore the component A_z of a gauge field A can be identified with the positive helicity mode, the component $A_{\bar{z}}$ with the negative helicity. The further non-vanishing components $(\epsilon^\pm)_-$ will play no role in the remaining discussion.

Twistor space. Ordinary space-time is the space on which the Poincare group acts linearly. In a Lagrangian it is therefore natural to integrate over this space in order to display the Poincare symmetry of the theory explicitly. Twistor space is the space on which the conformal group acts linearly. It is therefore a natural question whether one can formulate an action on this space directly, since it is known that Yang-Mills theory at the tree level enjoys a conformal invariance. Concretely, the question is whether there is a way to write a Lagrangian which involves an integration over twistor space instead of ordinary space-time. Since the Poincare group is a subgroup of the conformal group, it can be expected that this can be done.

For the purpose of the construction of actions on twistor space it is useful for technical reasons (see below) to work in Euclidean space instead of Minkowski space.¹ In this article we will be concerned with an off-shell version of the Penrose transform. For a review of on-shell twistor methods see [47]. The twistor space for flat four dimensional space-time is \mathbb{CP}^3 , with homogeneous coordinates Z ,

$$Z = \{(\omega^\alpha, \pi_{\dot{\alpha}}) \in \mathbb{C}^4 | Z \sim \lambda Z \forall \lambda \in \mathbb{C}^*\} \quad (2.11)$$

Here ω and π are two two-component spinors which transform naturally under two $SL(2, \mathbb{C})$ subgroups. Space-time arises as the space of holomorphic lines in this space by the so-called incidence relation

$$\omega^\alpha = X^{\alpha\dot{\alpha}} \pi_{\dot{\alpha}} \quad (2.12)$$

It is easy to verify using this equation that the natural action of the generator of the linear symmetry group of \mathbb{CP}^3 ,

$$J_{\bar{\beta}}^{\dot{\alpha}} \sim Z^{\dot{\alpha}} \frac{\delta}{\delta Z^{\bar{\beta}}}, \quad (2.13)$$

¹A Minkowski space version of the action follows by simple Wick rotation. More formally, one can work on the total space of the spin-bundle.

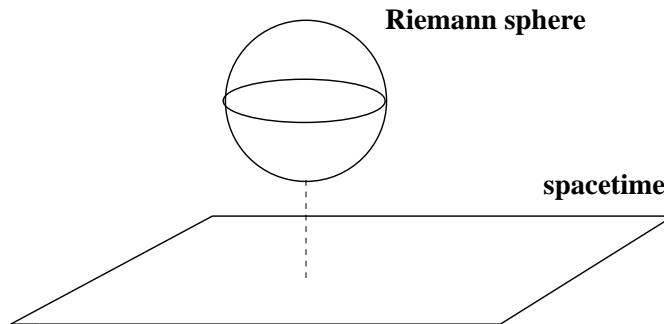


Figure 1: Twistor space in Euclidean signature is the total space of a \mathbb{CP}^1 fibre bundle

on the coordinates corresponds to the conformal group in four dimensional complexified space-time. This is of course a simple consequence of the complex algebra isomorphism $SU(4) \sim SO(6)$.

Solving the incidence relation for X requires a reality condition on this coordinate. In this article we will take Euclidean reality conditions, which are enforced by requiring reality under Euclidean conjugation denoted by hat,

$$\widehat{\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}} = \begin{pmatrix} -\bar{\pi}_2 \\ \bar{\pi}_1 \end{pmatrix}. \quad (2.14)$$

What is special about Euclidean signature is that the solution to (2.12) is actually unique in this case: the resulting equation for X is a projection map and \mathbb{CP}^3 can therefore be interpreted as the total space of a \mathbb{CP}^1 fibre bundle over R^4 . Therefore, it is useful in Euclidean signature to think of \mathbb{CP}^3 as $R^4 \times \mathbb{CP}^1$ with coordinates (x, π) as in figure 1. With the Euclidean conjugation we have the symmetric spinor products $\langle \pi \hat{\pi} \rangle = \langle \hat{\pi} \pi \rangle = \pi_1 \bar{\pi}_1 + \pi_2 \bar{\pi}_2$. We can then take $\kappa = \hat{\eta}$ in the decomposition (2.2) for the euclidean signature, resulting in the replacements $2 \rightarrow \langle \eta \hat{\eta} \rangle [\hat{\eta} \eta]$ and a plus sign in the expressions for p_z and $p_{\bar{z}}$ in (2.3).

The next step² in the twistor construction is the definition of the natural fields on the twistor space. This is most natural in terms of the homogeneous coordinates Z in \mathbb{C}^4 . In order for these fields to be well-defined on the projective space, they need to have definite weight under the scaling $Z \rightarrow \lambda Z$. Furthermore, a notion of 'holomorphicity' is very important on the twistor space. \mathbb{CP}^1 is one complex dimensional, so has a holomorphic and a anti-holomorphic direction. The R^4 is roughly \mathbb{C}^2 , so has two holomorphic and two anti-holomorphic directions. An 'anti-holomorphic vector' therefore has three components: two along space-time and one up the fibre direction. In more mathematical terms these 'anti-holomorphic vectors' are (dual to) $(0, 1)$ forms on the twistor space. It is useful from a space-time point of view to write the needed $(0, 1)$ forms in a basis with elements

$$\bar{e}_0 = \frac{\hat{\pi}^{\dot{\alpha}} d\hat{\pi}_{\dot{\alpha}}}{(\pi \hat{\pi})^2} \quad \bar{e}^\alpha = \frac{dx^{\alpha\dot{\alpha}} \hat{\pi}_{\dot{\alpha}}}{(\pi \hat{\pi})} \quad (2.15)$$

²We apologise to the mathematically inclined reader for the following slightly ham-fisted explanation of Dolbeault cohomology classes on twistor space.

Note that \bar{e}_0 points along the fibre, while the other 2 directions point along space-time. With the above basis of one-forms, anti-holomorphic one-form fields B can be expanded as

$$B = \bar{e}^\alpha B_\alpha + \bar{e}^0 B_0 \tag{2.16}$$

The B_α and B_0 therefore make up the components of what could be called a anti-holomorphic vector. Note that B_α and B_0 have holomorphic weight +1 and +2 respectively compared to the original weight of B with respect to scaling of the fibre coordinates. In addition, from a Lorentz point of view this (0, 1) form consists of a scalar and a spinor representation.

We will furthermore need a notion of anti-holomorphic vector derivation, denoted $\bar{\partial}$. As can be verified by contracting into the above choice of basis of (0, 1) forms its components are,

$$\bar{\partial}_0 = (\pi \cdot \hat{\pi}) \pi_{\dot{\alpha}} \frac{\partial}{\partial \hat{\pi}_{\dot{\alpha}}} \quad \bar{\partial}_\alpha = \pi^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha \dot{\alpha}}} \tag{2.17}$$

The most basic twistor theory result now equates Dolbeault cohomology classes on twistor space with solutions to the wave equation (see e.g. [47]). More concretely, if a (0, 1) form B of weight n satisfies

$$\bar{\partial} B = 0 \tag{2.18}$$

than it corresponds to a solution of the wave equation on space-time with helicity $\frac{n+2}{2}$. Note that this equation can be solved up to invariance $B \rightarrow B + \bar{\partial} f$ for some scalar field f , which we will interpret as a gauge symmetry. Identifying solutions up to this invariance is the definition of the cohomology class and yields a one-to-one map between this class and solutions to the wave-equation. Note that equation (2.18) in the basis introduced earlier splits into three equations

$$\begin{aligned} \bar{\partial}_0 B_\alpha - \bar{\partial}_\alpha B_0 &= 0 \\ \bar{\partial}_\alpha B^\alpha &= 0 \end{aligned} \tag{2.19}$$

In the appendix D it is shown explicitly in the case of a massless scalar how the solution to these equations up to gauge equivalence (i.e. the cohomology class) corresponds to a solution of the wave equation.

2.2 CSW rules from a canonical transformation

In this subsection we review the canonical transformation approach to the derivation of the CSW rules [27, 44] in order to prepare for the extension of this method to massive scalars given in section 3. The conventions used for the Yang-Mills fields are collected in appendix A. Expressing the gauge field in terms of light-cone components (2.2) and imposing the light cone gauge $A_+ = A \cdot \eta = 0$, the Yang-Mills Lagrangian depends only quadratically on the component A_- . The equation of motion reads

$$A_- = \frac{1}{\partial_+^2} ([D_z, \partial_+ A_{\bar{z}}] + [D_{\bar{z}}, \partial_+ A_z]) \tag{2.20}$$

where the covariant derivatives are $D_i = \partial_i - igA_i$ for $i = z, \bar{z}$. Using (2.20), the component A_- can be integrated out, resulting in a Lagrangian that contains only the two physical components of the gauge fields A_z (positive helicity) and $A_{\bar{z}}$ (negative helicity). The Lagrangian for the physical modes has the form

$$\mathcal{L}_A = \mathcal{L}_{A_z A_{\bar{z}}}^{(2)} + \mathcal{L}_{++-}^{(3)} + \mathcal{L}_{+--}^{(3)} + \mathcal{L}_{++--}^{(4)} \quad (2.21)$$

with the kinetic term

$$\mathcal{L}_{A_z A_{\bar{z}}}^{(2)} = -2 \text{tr} A_z \square A_{\bar{z}} = -4 \text{tr} A_z (\partial_+ \partial_- - \partial_z \partial_{\bar{z}}) A_{\bar{z}} \quad (2.22)$$

and the interaction terms

$$\mathcal{L}_{--+}^{(3)} = 4ig \text{tr} [[\partial_+ A_z, A_{\bar{z}}](\partial_+)^{-1}(\partial_z A_{\bar{z}})] \quad (2.23)$$

$$\mathcal{L}_{++-}^{(3)} = 4ig \text{tr} [[\partial_+ A_{\bar{z}}, A_z](\partial_+)^{-1}(\partial_{\bar{z}} A_z)] \quad (2.24)$$

$$\mathcal{L}_{++--}^{(4)} = 4(ig)^2 \text{tr} [[\partial_+ A_{\bar{z}}, A_z](\partial_+)^{-2}[\partial_+ A_z, A_{\bar{z}}]] \quad (2.25)$$

Note that for 'mostly +' CSW rules one would as a first step like to eliminate the non MHV-type coupling $\mathcal{L}_{++-}^{(3)}$. This vertex together with the kinetic term correspond [48] to the self-dual version of Yang-Mills theory. Eliminating this vertex therefore corresponds to solving self-dual Yang-Mills by mapping it to a free theory. This will be important when comparing to the twistor Yang-Mills formulation.

Concretely, the non-MHV vertex can be eliminated by a transformation to new variables B and \bar{B} [27] that satisfies the condition

$$\int d^3x [\mathcal{L}_{A_z A_z}^{(2)} + \mathcal{L}_{++-}^{(3)}] = \int d^3x \mathcal{L}_{\bar{B} B}^{(2)} \quad (2.26)$$

In order to have a unit Jacobian, the transformation of the fields and conjugate momenta ($A_z, \Pi_{A_z} \sim \partial_+ A_{\bar{z}}$) to new fields and momenta ($B, \Pi_B \sim \partial_+ \bar{B}$) has to be canonical. This requirement can be satisfied by a transformation with the generating functional of the form [27, 35]

$$G[A_z, \Pi_B] = \int d^3y B[A_z(\vec{y})] \Pi_B(\vec{y}) \quad (2.27)$$

Here x^- is treated as time variable common to all fields and the remaining coordinates are collected in the vector $\vec{x} = (x^+, x^z, x^{\bar{z}})$. The generating functional induces the transformations of the fields, $B = \delta G / \delta \Pi_B$, and of the momenta $\Pi_{A_z} = \delta G / \delta A_z$ resulting in the transformation

$$A_z \rightarrow B[A_z], \quad \partial_+ A_{\bar{z}}(x) = \int d^3y \frac{\delta B(x^-, \vec{y})}{\delta A_z(x^-, \vec{x})} \partial_+ \bar{B}(x^-, \vec{y}) \quad (2.28)$$

This transformation of the momentum guarantees that the structure of the kinetic term $A_{\bar{z}} \partial_+ \partial_- A_z$ remains intact, provided $B[A_z]$ depends on x^- only implicitly through A_z . Using the form of the transformation (2.28), the condition for the elimination of the non-MHV vertex (2.26) can be written as [27]

$$\int d^3y \left(\frac{\partial_z \partial_{\bar{z}}}{\partial_+} A_z(\vec{y}) - ig \left[A_z(\vec{y}), \frac{\partial_{\bar{z}}}{\partial_+} A_z(\vec{y}) \right] \right) \frac{\delta B(\vec{x})}{\delta A_z(\vec{y})} = \frac{\partial_z \partial_{\bar{z}}}{\partial_+} B(\vec{x}) \quad (2.29)$$

The dependence on the common light-cone time x_- has been left implicit. In momentum space, the equations for A_z and $\partial_+ A_{\bar{z}}$ can be solved using the Ansätze

$$A_{p,z} = \sum_{n=1}^{\infty} \int \prod_{i=1}^n \widetilde{dk}_i \mathcal{Y}(p, k_1, \dots, k_n) B_{-k_1} \dots B_{-k_n} \quad (2.30)$$

$$p_+ A_{p,\bar{z}} = \sum_{n=1}^{\infty} \sum_{s=1}^n \int \prod_{i=1}^n \widetilde{dk}_i \mathcal{X}^s(p, k_1, \dots, k_n) B_{-k_1} \dots (k_{s+}) \bar{B}_{-k_s} \dots B_{-k_n} \quad (2.31)$$

with the integration measure $\widetilde{dp} = dp_+ dp_z dp_{\bar{z}} / (2\pi)^3$ and keeping a momentum conserving delta-function $(2\pi)^3 \delta^3(p + \sum_i k_i)$ implicit.

The condition (2.29) can be used to extract a recursion relation for the coefficients \mathcal{Y} [44]:

$$\omega_{1,n} \mathcal{Y}(k_1, \dots, k_n) = \sum_{j=2}^{n-1} g(\zeta_{k_{j+1},n} - \zeta_{k_{2,j}}) \mathcal{Y}(-k_{2,j}, k_2, \dots, k_j) \mathcal{Y}(-k_{j+1,n}, k_{j+1}, \dots, k_n) \quad (2.32)$$

where a number of abbreviations have been defined as follows:

$$k_{i,j} = \sum_{\ell=i}^j k_{\ell} \quad , \quad \omega_p = \frac{p_{\bar{z}} p_z}{p_+} \quad , \quad \omega_{i,j} = \sum_{k=i}^j \omega_k \quad , \quad \zeta_p = \frac{p_{\bar{z}}}{p_+} \quad (2.33)$$

Note that the prefactor $(\zeta_{k_{j+1},n} - \zeta_{k_{2,j}})$ appearing in (2.32) is proportional to the momentum-space vertex arising from the interaction $\mathcal{L}_{++-}^{(3)}$ that the canonical transformation is designed to eliminate [32]. This recursion relation is similar to the Berends-Giele relation for the one-particle off-shell current with positive helicity gluons [2, 39] and can be solved with similar methods. The difference is that in the present case all gluons can be off-shell, in contrast to the case of the Berends-Giele relation.

The solution of the conditions (2.32) and (2.28) has been found by Eittle and Morris (EM) [44] and reads in our conventions

$$\mathcal{Y}(p, k_1, \dots, k_n) = (-ig)^{n-1} \frac{i^{n+1} p_+ k_{2+} \dots k_{(n-1)+}}{(1,2) \dots ((n-1),n)} = \frac{(g\sqrt{2})^{n-1} \langle \eta p \rangle^2}{\langle \eta 1 \rangle \langle 12 \rangle \dots \langle (n-1)n \rangle \langle n\eta \rangle} \quad (2.34)$$

$$\mathcal{X}^s(p, k_1, \dots, k_n) = -\frac{k_{\pm}^s}{p_+} \mathcal{Y}(p, k_1, \dots, k_n) = -\frac{(g\sqrt{2})^{n-1} \langle \eta s \rangle^2}{\langle \eta 1 \rangle \langle 12 \rangle \dots \langle (n-1)n \rangle \langle n\eta \rangle} \quad (2.35)$$

Inserting the expressions (2.30) and (2.31) into the light-cone Lagrangian (2.21) leads to a tower of vertices that contain two \bar{B} fields and an arbitrary number of B fields:

$$\mathcal{L} = \mathcal{L}_{B\bar{B}}^{(2)} + \sum_{n=3}^{\infty} \mathcal{L}_{+\dots+--}^{(n)} \quad (2.36)$$

The interaction vertices in momentum space take the form

$$\mathcal{L}_{+\dots+--}^{(n)} = \frac{1}{2} \sum_{j=2}^n \int \prod_{i=1}^n \widetilde{dk}_i \mathcal{V}_{\bar{B}_1, B_2, \dots, \bar{B}_j, \dots, B_n} \text{tr}(\bar{B}_{k_1} \dots \bar{B}_{k_j} \dots B_{k_n}) \quad (2.37)$$

It has been argued [27] and explicitly checked up to $n = 5$ [44] that the coefficients $\mathcal{V}_{\bar{B}_1, \dots, \bar{B}_i, \dots, B_n}$ are just the MHV amplitudes continued off-shell according to the CSW prescription.

Since the field-redefinitions (2.30) and (2.31) begin with a linear term, i.e. $A_z = B + \mathcal{O}(B)^2$ and $A_{\bar{z}} = \bar{B} + \mathcal{O}(\bar{B}B)$, standard reasoning (see e.g. [49]) suggests that on-shell matrix elements of the new and old fields coincide up to a possible wave-function renormalisation (this has been shown to be absent at one loop [44]) since the non-linear terms naively do not contribute to single particle poles. This result is sometimes known as the 'equivalence theorem'. However, due to the non-local character of the field-redefinition which violates one of the assumptions in the theorem the situation is more subtle. This will be discussed in more detail in 2.5.

2.3 CSW rules from twistor Yang-Mills theory

As stated before, the Penrose correspondence relates Dolbeault cohomology classes for $(0, 1)$ forms of specified weight with solutions to the wave equation on space-time with specified helicity. This means that the most natural action on twistor space is first order. Based on this information and gauge-invariance we will review the construction of an action on twistor space from the ordinary space-time one by means of an off-shell version of the Penrose transform.

The basic building block is a $(0, 1)$ form B . Interpreting this as a gauge field, three natural independent $(0, 2)$ form curvatures can be defined by commuting covariant derivatives, $F_{\alpha\beta} = \epsilon_{\alpha\beta}F$ and $F_{0\alpha}$. Since we are only interested in curvature along space-time, we will impose $F_{0\alpha} = 0$. Note the close connection of this constraint to the constraints used in harmonic superspace methods [50]. From this one can derive [30] the 'lifting' formula

$$-igA_{\alpha\dot{\alpha}}(x)\pi^{\dot{\alpha}} = H^{-1} \left(\bar{\partial}_{\alpha} - \sqrt{2}igB_{\alpha} \right) H \tag{2.38}$$

Here H are holomorphic frames such that $(\bar{\partial}_0 - i\sqrt{2}gB_0)H = 0$. These frames are defined at each space-time point x and always exists perturbatively since Yang-Mills theory on the 'extra' two dimensional sphere $\mathbb{CP}^1 = S^2$ has no local degrees of freedom. Working out the remaining curvature, one arrives at the intriguing formula

$$\begin{aligned} \frac{1}{2}H^{-1} \left[\bar{\partial}_{\alpha} - i\sqrt{2}gB_{\alpha}, \bar{\partial}^{\alpha} - i\sqrt{2}gB^{\alpha} \right] H &= -i\sqrt{2}gH^{-1} \left(\bar{\partial}_{\alpha}B^{\alpha} - i\sqrt{2}gB_{\alpha}B^{\alpha} \right) H \\ &= -igF_{\dot{\alpha}\dot{\beta}}(x)\pi^{\dot{\alpha}}\pi^{\dot{\beta}} \end{aligned} \tag{2.39}$$

where $F_{\dot{\alpha}\dot{\beta}}$ is the self-dual part of the space-time curvature. As stated before, the most natural action on twistor space is first order. With this fact and given equation (2.39), it is natural to study the Chalmers and Siegel action [51],

$$S_{\text{Chalmers and Siegel}} = \text{tr} \int dx^4 C_{\dot{\alpha}\dot{\beta}} F_{+}^{\dot{\alpha}\dot{\beta}} - \frac{1}{2} C_{\dot{\alpha}\dot{\beta}} C^{\dot{\alpha}\dot{\beta}}. \tag{2.40}$$

Here C is an auxiliary self-dual two-form. Demanding that the first term is local on twistor space then yields the lifting formula for the self-dual field C ,

$$C_{\dot{\alpha}\dot{\beta}} = \sqrt{2} \int_{\mathbb{CP}^1} dk H^{-1} \bar{B}_0 H \pi_{\dot{\alpha}} \pi_{\dot{\beta}} \tag{2.41}$$

Here dk is the natural volume form on \mathbb{CP}^1 ,

$$dk = \frac{\langle \pi d\pi \rangle \langle \hat{\pi} d\hat{\pi} \rangle}{\langle \pi \hat{\pi} \rangle} \quad (2.42)$$

Note that the above transform for C can also be arrived at by more carefully considering [45] the C^2 term: since this will be non-local on twistor space, to keep gauge invariance a gauge link operator between the two operators has to be inserted. This Wilson line operator is naturally formed by the frames and because it is a link in 2 dimensions it is independent of the path taken.

Putting in the transforms and the constraint $F_{0\alpha} = 0$ by Lagrange multiplier then yields the full twistor action for Yang-Mills theory

$$S = 2 \operatorname{tr} \int d^4x dk \bar{B}_0 \left(\bar{\partial}^\alpha B_\alpha - i\sqrt{2}g B^\alpha B_\alpha \right) + \bar{B}^\alpha \left(\bar{\partial}_\alpha B_0 - \bar{\partial}_0 B_\alpha - i\sqrt{2}g [B_\alpha, B_0] \right) \\ - \operatorname{tr} \int d^4x dk_1 dk_2 H_1^{-1} \bar{B}^0(\pi_1) H_1 H_2^{-1} \bar{B}^0(\pi_2) H_2 \langle \pi_1 \pi_2 \rangle^2 \quad (2.43)$$

This action is invariant under $B \rightarrow B + \bar{\partial}_B f^0$, $\bar{B} \rightarrow \bar{B} - i\sqrt{2}g[\bar{B}, f^0]$, as well as $\bar{B} \rightarrow \bar{B} + \bar{\partial}_B f^{-4}$ for two independent gauge parameters of the indicated weight and the covariant derivative $\bar{\partial}_B = \bar{\partial} - ig\sqrt{2}B$. Note that the second symmetry arises from the usual 'quantising with constraints' procedure. It is therefore natural to interpret \bar{B} as a second $(0, 1)$ form, and the action can be written as

$$S = \operatorname{tr} \int d^4x dk \bar{B} \wedge (\bar{\partial}_B B) \\ - \operatorname{tr} \int d^4x dk_1 dk_2 \langle \pi_1 \pi_2 \rangle^2 (H^{-1} \bar{B} H)_{(1)} \wedge (H^{-1} \bar{B} H)_{(2)} \quad (2.44)$$

Since both gauge symmetry parameters live on the (six real dimensional) twistor space, the gauge symmetry is *larger* than on space-time. In order to calculate correlation functions for scattering amplitudes from this action, it needs to be fixed and two gauge choices will be discussed below.

The action derived here can also be derived from the original twistor string proposal [4] by truncating to single trace terms, as is in principle clear already from [5]. As a historical aside, note that the same action follows from simply adding the result on MHV amplitudes in [3] to the local twistor space action of [52].

Space-time gauge. Since the part of the gauge freedom which is not present on space-time resides on the \mathbb{CP}^1 fibre, it is natural to try to fix this extra symmetry with a Lorenz-like gauge

$$\bar{\partial}^\dagger B_0 = \bar{\partial}^\dagger \bar{B}_0 = 0 \quad (2.45)$$

Working out this gauge choice turns out to reduce the action in (2.43) back to ordinary Yang-Mills [33]. In particular the infinite tower of interactions in the second term reduces to a quadratic term. The ghosts associated to this gauge condition can be shown to decouple. The residual gauge symmetry left from the above gauge-fixing is exactly ordinary space-time gauge symmetry. Fixing this will, of course, lead directly to ordinary Yang-Mills perturbation theory.

CSW gauge. More general gauges are possible. Picking an arbitrary spinor η_α , the CSW gauge is the axial gauge choice

$$\eta_\alpha B^\alpha = \eta_\alpha \bar{B}^\alpha = 0 \tag{2.46}$$

An important consequence of this choice is that it eliminates the vertex in the 'CF' part of the action. Since this term came from the interaction in F^+ , this gauge choice eliminates the original 'self-dual' vertex. The propagators and on-shell fields can be calculated by an ordinary Fourier transform on the space-time coordinate. This changes coordinates,

$$\{x_{\alpha\dot{\alpha}}, \pi_{\dot{\beta}}\} \rightarrow \{ip_{\alpha\dot{\alpha}}, \pi_{\dot{\beta}}\}. \tag{2.47}$$

The propagator relevant here³ is

$$: B_0 \bar{B}_0 := \frac{i\delta(\eta\pi_1 p)\delta(\eta\pi_2 p)}{p^2}. \tag{2.48}$$

The only vertices left in the theory are now the infinite tower in the part which originated from C^2 . These already have the right shape to be the MHV scattering amplitudes, as an expansion in terms of the fields gives,

$$C^2 = -\text{tr} \sum_{n=2}^{\infty} (-i\sqrt{2}g)^{n-2} \int \sum_{p=2}^n \frac{\langle \pi_1 \pi_p \rangle^4}{\langle \pi_1 \pi_2 \rangle \langle \pi_2 \pi_3 \rangle \dots \langle \pi_n \pi_1 \rangle} \text{tr} \left(\bar{B}_0^1 B_0^2 \dots B_0^{p-1} \bar{B}_0^p B_0^{p+1} \dots B_0^n \right) \tag{2.49}$$

To calculate the scattering amplitudes correctly, one needs the relation between correlators and scattering amplitudes given by LSZ. This can be worked out [28] for the twistor action starting from ordinary Yang-Mills using 2.38. There is a subtlety here related to the application of the 'equivalence theorem' which will be addressed below in section 2.5. With this taken into account, the remaining vertices generate exactly MHV amplitudes, and from there it is a small step to see the complete CSW rules. Note how the CSW spinor arises here from the gauge fixing spinor, and the independence of this spinor needed for Lorentz invariance translates therefore to gauge invariance.

One should note the close similarity of the above gauge choices with harmonic superspace methods. Harmonic superspace has a similar extended gauge symmetry and basically the same gauge choices are used. The main difference is that where the 'CSW' gauge here breaks one of the $SL(2, \mathbb{R})$ Lorentz subgroups, in the harmonic superspace case the corresponding gauge only breaks the $SU(2)$ R -symmetry.

2.4 Comparison of the two approaches

Above we reviewed two seemingly very different action-based derivations of the CSW rules. A natural question is how these two are related. On the level of perturbation theory it is clear that the two approaches will be equivalent since both lead to the same expressions for Feynman diagrams. They are also based on the same principle: both methods trivialise the self-dual Yang-Mills equations by a canonical field transformation. However, this does not

³There are more propagators, but these can be shown to be irrelevant for calculating amplitudes because the only vertex in the game contains only B_0 and \bar{B}_0 fields.

show that the lightcone approach to MHV diagrams is the same as the twistor one on the level of the amplitudes. The remaining gap is that the canonical transformation method has only been checked to give off-shell MHV vertices up to 5 gluons, although there is no reason to really doubt the general result. In the twistor-action, on the other hand, the all-multiplicity vertices can be easily derived but the action is formulated in terms of fields on twistor space instead of space time. One of the ways to complete the argument on the level of the action is to start with the twistor action, impose CSW gauge and integrate out all the components of the gauge fields which do not appear in the actual rules. That this can be done in principle is clear. However, one encounters some obstacles as, for instance, the conjugate component to $\eta\eta A$ is a non-linear combination of twistor fields. On the other hand, the field equation for the component of \bar{B} which is gauged to zero will impose some constraint which need to be taken into account. A second approach would be to take the Chalmers and Siegel action and apply the canonical transformation method directly to this.

Instead of attempting to show equivalence on the level of the action, here we relate the field transformations used in the two approaches and show that the canonical transformation is the space-time interpretation of the twistor gauge transformation. This implies that the twistor space gauge symmetry is the linear version of the non-linear symmetry probed by the canonical transformation. Some evidence that this is true was uncovered in [30]; below we complete the argument.

The space-time interpretation of the twistor gauge transformation utilised above to derive the original CSW rules follows directly from the lifting formula for the gauge field (2.38). In CSW gauge we have,

$$-ig\eta^\alpha A_{\alpha\dot{\alpha}}(x)\pi^{\dot{\alpha}} = \eta^\alpha H^{-1}(\pi) (\bar{\partial}_\alpha) H(\pi) \quad (2.50)$$

which is almost an axial gauge on space-time. This observation can be made precise by considering the frames H : these are the solutions to $(\bar{\partial}_0 - i\sqrt{2}gB_0)H = 0$. For a definite solution of this differential equation, a boundary condition must be chosen. If one chooses $H(\eta) = 1$ for an arbitrary point $\eta^{\dot{\alpha}}$,

$$\eta^\alpha \eta^{\dot{\alpha}} A_{\alpha\dot{\alpha}} = 0 \quad (2.51)$$

holds. Hence the space-time interpretation of the twistor CSW gauge is light-cone gauge, with $\eta^\alpha \eta^{\dot{\alpha}}$ the gauge-fixing vector. The main advantage of light-cone gauge on space-time is that the physical polarisations of the gluon are manifest and are given in terms of twistor fields by:

$$A_z \equiv \frac{1}{2}\eta^\alpha \hat{\eta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x) = \frac{i}{2g} H^{-1}(\hat{\eta}) (\eta^\alpha \hat{\eta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}}) H(\hat{\eta}) \quad (2.52)$$

and

$$A_{\bar{z}} = \frac{1}{2}\hat{\eta}^\alpha \eta^{\dot{\alpha}} A_{\alpha\dot{\alpha}}(x) = \frac{\sqrt{2}}{2}\hat{\eta}^\alpha B_\alpha(\eta) \quad (2.53)$$

The \bar{B}_0 field equation from the action (2.43) can then be used to give for the second equality

$$i(\eta_\alpha \eta_{\dot{\alpha}} p^{\alpha\dot{\alpha}}) A_{\bar{z}}(p) = \frac{1}{\sqrt{2}}(\hat{\eta}^\alpha \eta_\alpha) \int_{\mathbb{CP}^1} dk H(\eta) H^{-1} \bar{B}_0 H H^{-1}(\eta) \langle \pi \eta \rangle^2 \quad (2.54)$$

Let us focus on equation (2.52) first. The frames can be expanded into B_0 fields using

$$H(\hat{\eta}) = H(\hat{\eta})H^{-1}(\eta) = \frac{H(\hat{\eta})H^{-1}(\eta)}{\langle \hat{\eta}\eta \rangle} \langle \hat{\eta}\eta \rangle = \left(\frac{1}{\bar{\partial}_0 - i\sqrt{2}gB_0} \right)_{\hat{\eta}\eta} \langle \hat{\eta}\eta \rangle \quad (2.55)$$

Note that this formula is local on space-time. Now

$$\delta \left(\frac{1}{\bar{\partial}_0 - i\sqrt{2}gB_0} \right)_{\hat{\eta}\eta} = i\sqrt{2}g \int_{\pi_k} \frac{H(\hat{\eta})H^{-1}(\pi_k)}{\langle \hat{\eta}\pi_k \rangle} \delta B_0 \frac{H(\pi_k)H^{-1}(\eta)}{\langle \pi_k\eta \rangle} \quad (2.56)$$

can be used to find

$$A_z(x) = -\frac{1}{\sqrt{2}} \langle \hat{\eta}\eta \rangle \int_{\pi_k} \frac{H(\eta)H^{-1}(\pi_k)}{\langle \eta\pi_k \rangle} \left(\frac{\langle \eta\pi_k \rangle}{\langle \hat{\eta}\pi_k \rangle} \eta^\alpha \hat{\eta}^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} B_0 \right) \frac{H(\pi_k)H^{-1}(\eta)}{\langle \pi_k\eta \rangle} \quad (2.57)$$

This expression can be expanded into components. Observe that when calculating scattering amplitudes through Feynman diagrams in the twistor action approach, all the B_0 fields in the above expression will only be contracted with \bar{B}_0 fields, leading to delta-functions of $(\eta_\alpha \pi_{\dot{\alpha}} p^{\alpha\dot{\alpha}}) \equiv \langle \pi(\eta p) \rangle$ on the fibre. To see what the above expression corresponds to on space-time we can therefore insert the delta functions everywhere and perform the integrals. Expanding to third order for instance yields

$$A_z(p) = \frac{-i}{\sqrt{2}} \langle \hat{\eta}\eta \rangle \left(\frac{B_0(q_1)}{\langle \eta(\eta q_1) \rangle} - ig \int_{q_1, q_2} \langle \eta(\eta p) \rangle \frac{B_0(q_1)B_0(q_2)}{\langle \eta(\eta q_1) \rangle \langle (\eta q_1)(\eta q_2) \rangle \langle \eta(\eta q_2) \rangle} \right. \\ \left. - \sqrt{2}g^2 \int_{q_1, q_2, q_3} \langle \eta(\eta p) \rangle \frac{B_0(q_1)B_0(q_2)B_0(q_3)}{\langle \eta(\eta q_1) \rangle \langle (\eta q_1)(\eta q_2) \rangle \langle (\eta q_2)(\eta q_3) \rangle \langle \eta(\eta q_3) \rangle} + \dots \right) \quad (2.58)$$

Here momentum conservation was used to sum the numerator. The integral shorthand is defined by $\int_{q_1, \dots, q_n} = \int d^4q_1 \dots d^4q_n \delta^4(p + q_1 + \dots + q_n)$. Proceeding further shows that the full sum will be exactly the solution to the recursion relations in equation (2.35) originally found by Eittle and Morris [44]. To see this more explicitly, note first that the linear term in this expansion shows that one has to apply a normalisation factor $\langle \eta(\eta q_i) \rangle$ for each external B_0 leg in order to obtain the correctly normalised A_z correlation functions. Furthermore the expression for the EM-coefficients (2.35) assumes that the external wavefunction normalisation is trivial. Therefore through inserting the correct pre-factor for the polarisation factors (2.9) the field B_0 in the twistor approach is related to the field B in the canonical approach by a normalisation factor

$$B(q_i) \Rightarrow \frac{i}{\sqrt{2}} \frac{B_0(q_i)}{\langle \eta(\eta q_i) \rangle (\epsilon^+(q_i))_z} = -i \frac{B_0(q_i)}{[\eta i]^2} \quad (2.59)$$

where the spinor products in the last term are continued off-shell using (2.8). Indeed, inserting the external polarisation factor of A_z into (2.58) precisely reproduces the expression (2.35) for the EM-coefficients for the correctly normalised fields:

$$(\epsilon^+(p))_z^{-1} A_z(p) = \frac{-iB_0(q_1)}{[\eta 1]^2} + \int_{q_1, q_2} (\sqrt{2}g) \frac{\langle \eta p \rangle^2}{\langle \eta 1 \rangle \langle 12 \rangle \langle 2\eta \rangle} \frac{(-iB_0(q_1))}{[\eta 1]^2} \frac{(-iB_0(q_2))}{[\eta 2]^2} \\ + \int_{q_1, q_2, q_3} (\sqrt{2}g)^2 \frac{\langle \eta p \rangle^2}{\langle \eta 1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3\eta \rangle} \frac{(-iB_0(q_1))}{[\eta 1]^2} \frac{(-iB_0(q_2))}{[\eta 2]^2} \frac{(-iB_0(q_3))}{[\eta 3]^2} + \dots \quad (2.60)$$

Note that the denominator of that expression simply follows here from the general structure of the lifting formula (2.38) in terms of link operators. Furthermore, note that all the momenta in the above expression do not depend on the momentum-component $\hat{\eta}^\alpha \hat{\eta}^{\dot{\alpha}} p_{\alpha\dot{\alpha}}$, so the momentum integrals along this component of the momenta can all be performed trivially. This explains the difference between the 4 dimensional momentum integrals utilised above and the 3 dimensional momentum integrations natural in lightcone Yang-Mills.

The other component can be treated using similar techniques, as we can write

$$i(\eta\eta p)A_{\bar{z}}(p) = -\frac{1}{\sqrt{2}}(\hat{\eta}^\alpha \eta_\alpha) \int_{\pi_k} dk \frac{H(\eta)H^{-1}(\pi_k)}{\langle \eta\pi_k \rangle} \bar{B}_0(\pi_k) (\langle \pi_k \eta \rangle)^4 \frac{H(\pi_k)H^{-1}(\eta)}{\langle \pi_k \eta \rangle} \quad (2.61)$$

Expanding and evaluating will yield the same coefficients as found in the canonical transformation case. The field \bar{B}_0 will appear in the combination

$$\sim i[\eta 1]^2 \bar{B}_0(q_1) \quad (2.62)$$

which again is a consequence of the polarisation vector for \bar{B}_0 . In more detail, expanding the above equation yields,

$$(\epsilon^-(p))_{\bar{z}}^{-1}(\eta\eta p)A_{\bar{z}}(p) = i \frac{[\eta p]}{\langle \eta p \rangle} \left(\bar{B}_1[\eta 1]^2 \langle \eta 1 \rangle^2 - i\sqrt{2g} \int_{q_1, q_2} \left(\frac{\langle \eta 1 \rangle^3}{\langle 12 \rangle \langle 2\eta \rangle} \frac{[\eta 1]^2 \bar{B}_1 B_2}{[\eta 2]^2} + \frac{\langle \eta 2 \rangle^3}{\langle \eta 1 \rangle \langle 12 \rangle} \frac{B_1 \bar{B}_2 [\eta 2]^2}{[\eta 1]^2} \right) + \dots \right) \quad (2.63)$$

The difference to (2.31) is fully contained in the convention of (2.7). In particular, this makes the prefactor in the above equation disappear. The other change needed to see the equivalence is

$$\langle \eta p_B \rangle^2 \rightarrow (k_s)_+ \quad (2.64)$$

Note that in order to derive equation 2.61, the field equation for \bar{B}_0 was used, which will be important later. We like to stress that above we have used the fact that B_0 and \bar{B}_0 are always accompanied by a δ function on the fibres in this gauge and the above identification should be understood in that way. By the above calculation, in this sense it is shown that the canonical transformation coefficients can be derived from the twistor lifting formulae.

In our view the above calculation clearly identifies the non-local and non-linear canonical transformation employed in the 'canonical' approach to MHV diagrams with a local and linear gauge transformation on twistor space. In the following we might sometimes illustrate a point in one formalism, but by the above argument the same point could at least in principle be made in the other. Both have advantages and disadvantages. The canonical approach is calculational more intense and does not easily yield the full MHV Lagrangian, but is conceptually transparent. The twistor action approach requires some mathematical insight and requires one to choose the 'right' first order space-time action to lift,⁴ but makes

⁴Applying lifting formulae to F^2 directly for instance does not immediately lead to nice results!

the underlying symmetry and geometry transparent and as such immediately yields complete Lagrangians. These advantages and disadvantages manifest themselves for instance in the range of applicability of both methods: the canonical method has been applied to Einstein gravity with some success for the four point interaction [35], where the same has not been done yet through a twistor action. In contrast, the twistor action has already yielded the complete Nair super-vertex [33], where this has only been derived up to four point vertices [34] in the canonical approach.

2.5 Equivalence theorem violations

The results discussed in the section so far leave two questions open. The first is the three-point googly-MHV vertex

$$V(A_{k_1}(+), A_{k_2}(+), A_{k_3}(-)) = i\sqrt{2} \frac{[21]^3}{[32][13]} = i\sqrt{2}[21] \frac{\langle 3\eta \rangle^2}{\langle 1\eta \rangle \langle 2\eta \rangle} \quad (2.65)$$

that has been transformed away through the canonical transformation or the choice of CSW gauge in the twistor Yang-Mills action. This vertex controls a vanishing amplitude on-shell for real momenta in Minkowski signature but this amplitude is non-vanishing for complex momenta or for Euclidean signature [4]. The other missing pieces in pure Yang-Mills are the rational parts of one-loop amplitudes, including the purely rational amplitudes with less than two negative helicity gluons. One resolution noted in the literature requires to relate carefully the correlation functions of the original fields ($A_z, A_{\bar{z}}$) to that of the new fields (B, \bar{B}) [32]. Here we will review the argument for the recovery of the googly three particle MHV amplitude.

Using the original field variables, the vertex (2.65) is obtained by the LSZ formula from a correlation function of A fields as

$$\mathcal{A}_3(A_{k_1}(+), A_{k_2}(+), A_{k_3}(-)) = \prod_{i=1}^3 \lim_{k_i^2 \rightarrow 0} (-ik_i^2) \langle 0 | (\epsilon_{k_1}^+ \cdot A_{k_1}) (\epsilon_{k_2}^+ \cdot A_{k_2}) (\epsilon_{k_3}^- \cdot A_{k_3}) | 0 \rangle \quad (2.66)$$

Evaluating the right-hand side in the light-cone gauge using $(\epsilon^\pm \cdot A) = -(\epsilon^\pm)_{z/\bar{z}} A_{\bar{z}/z}$ leads to consider the Green's function $\langle 0 | A_{k_1, \bar{z}} A_{k_2, \bar{z}} A_{k_3, z} | 0 \rangle$. Note that this still contains propagators connecting A_z and $A_{\bar{z}}$ fields. To derive the same amplitude from the MHV Lagrangian one considers the Green's function as a function of the new fields by inserting the field-redefinitions (2.30) and (2.31) or equivalently the relations to the twistor-fields (2.52) and (2.53):

$$\begin{aligned} \langle 0 | A_{p_1, \bar{z}} A_{p_2, \bar{z}} A_{p_3, z} | 0 \rangle = & \left\langle 0 | \bar{B}_{p_1} \bar{B}_{p_2} B_{p_3} + \int \widetilde{dk_1} \widetilde{dk_2} \left[\left(\frac{k_{1+}}{p_{1+}} \mathcal{X}^1(p_1, k_1, k_2) \bar{B}_{-k_1} B_{-k_2} \right) \bar{B}_{p_2} B_{p_3} \right. \right. \\ & + \bar{B}_{p_1} \left(\frac{k_{2+}}{p_{2+}} \mathcal{X}^2(p_2, k_1, k_2) B_{-k_1} \bar{B}_{-k_2} \right) B_{p_3} \\ & \left. \left. + \bar{B}_{p_1} \bar{B}_{p_2} (\mathcal{Y}(p_3, k_1, k_2) B_{-k_1} B_{-k_2}) \right] + \dots | 0 \right\rangle \quad (2.67) \end{aligned}$$

where two more contributions in the expansion of the $A_{\bar{z}}$ fields have not been shown. Since there is no vertex for a single \bar{B} field in the MHV Lagrangian, the first term vanishes.

Usually the non-linear terms in the field redefinitions are argued not to contribute to the on-shell scattering amplitudes after the LSZ reduction since higher order expansions at a given site will generically lead to multi-particle poles and not to the single particle poles required for non-trivial LSZ contributions. This is the content of the so-called 'equivalence theorem' (see e.g. the discussion in [49]). In [32] it was shown that even at tree level one needs to be careful in applying the theorem: the non-local nature of the canonical transformation violates one of the assumptions, and summing seemingly vanishing contributions can lead to non-vanishing amplitudes.

A quick (admittedly slightly non-rigorous) way of finding the missing 3 point amplitude is to take asymmetric on-shell limits in the LSZ procedure. Take for instance the LSZ reduction (and the on-shell limit) on site 1 and 2, leaving site 3 off-shell. Then only the last contribution will survive. Now note that for $p_1^2 = p_2^2 = 0$ the three-point EM-coefficients are simply proportional to the three-point googly-MHV vertex:

$$\mathcal{Y}(p_3, p_1, p_2) = g\sqrt{2} \frac{\langle \eta 3 \rangle^2}{\langle \eta 1 \rangle \langle 12 \rangle \langle 2\eta \rangle} = g \frac{i}{\langle 12 \rangle [21]} V(A_{p_3, \bar{z}}, A_{p_1, z}, A_{p_2, z}) \quad (2.68)$$

Since legs 1 and 2 are on-shell, the denominator becomes $\langle 12 \rangle [21] = 2(p_1 \cdot p_2) = p_3^2$ so despite appearances the last term in (2.67) has a pole in p_3^2 . Performing now the LSZ reduction on this leg gives a non-vanishing result and leads to the correct three-point amplitude. One can check that the same result is reached by a different order of these asymmetrical LSZ-limits.

To see why these asymmetric LSZ-limits work, consider the case that all particles are off-shell. Using (2.8) one can express the denominator in (2.68) as

$$\begin{aligned} \langle 12 \rangle [21] &= \frac{\langle \eta + \not{p}_1 \not{p}_2 \not{p}_1 | \eta + \rangle}{2(p_1 \cdot \eta)} - 2p_2^2 \frac{(p_1 \cdot \eta)}{(p_2 \cdot \eta)} \\ &= -\frac{1}{(p_1 \cdot \eta)(\eta \cdot p_2)} [p_1^2(\eta \cdot p_2)^2 + p_2^2(p_1 \cdot \eta)^2 - 2(p_1 \cdot p_2)(\eta \cdot p_1)(\eta \cdot p_2)] \end{aligned} \quad (2.69)$$

Applying (2.69) e.g. to the denominator in (2.68) one sees that there is no pole in p_3^2 as long as p_1 and p_2 are not *both* on-shell so this term doesn't contribute if the LSZ reduction is performed for, say, legs 2 and 3 first. However, as demonstrated in [32], if all legs are initially kept off-shell the sum of the three terms in (2.67) survives the LSZ reduction although the separate terms don't contain three single-particle poles explicitly.

Note that only for the three-point case the right hand side of (2.69) can be expressed entirely in terms of the squares of the external momenta and a pole in p_3^2 can develop for $p_1^2 = p_2^2 \rightarrow 0$. This indicates that, at tree level, 'equivalence theorem evasion' is only expected to play a role in the three particle case. In this case the two legs connected to the 'vertex' must be on-shell (and null) to give a contributing pole. This is analogous to the non-vanishing of the three-point non-MHV vertex for complex kinematics [4, 23] while the one-particle off-shell currents with one negative helicity gluon and more than two positive helicity ones [2, 39] vanish upon LSZ reduction, also for complex kinematics.

3. CSW rules for massive scalars

3.1 Summary of the rules

In this section we summarise the CSW rules for scalar matter coupled to glue [37] and present further examples of their applications. These rules hold for a scalar ϕ in the fundamental representation of a $U(N)$ gauge group⁵ described by the space-time Lagrangian

$$\mathcal{L} = \mathcal{L}(A) + \mathcal{L}_\phi = \mathcal{L}(A) + (D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi \quad (3.1)$$

where $D_\mu = \partial_\mu - igA$. Some further explicit examples of the application of the CSW rules derived from this Lagrangian at tree level are given in 3.2. In sections 3.3 and 3.4 we offer two detailed derivations of CSW-like rules starting from the Lagrangian (3.1), based on the canonical transformation in the light-cone gauge and on the twistor Yang-Mills approach, respectively.

In both approaches used to derive the CSW rules a field-redefinition $\phi \rightarrow \xi$ of the scalars is performed that "dresses" them with an infinite number of positive helicity gluons and eliminates the non-MHV coupling of two scalars and a positive helicity gluon. The key idea in both derivations is to perform the *same* transformation of the massive scalars as in the massless case. In this way we arrive at a formalism that includes roughly the same vertices as the CSW formalism for massless scalars [6, 7] derived by supersymmetry arguments:

$$V_{\text{CSW}}(\bar{B}_1, B_2, \dots, \bar{B}_i, \dots, B_n) = i2^{n/2-1} \frac{\langle 1i \rangle^4}{\langle 12 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle} \quad (3.2)$$

$$V_{\text{CSW}}(\bar{\xi}_1, B_2, \dots, \bar{B}_i, \dots, \xi_n) = -i2^{n/2-1} \frac{\langle in \rangle^2 \langle 1i \rangle^2}{\langle 12 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle} \quad (3.3)$$

$$V_{\text{CSW}}(\bar{\xi}_1, B_2, \dots, \xi_i, \bar{\xi}_{i+1}, \dots, \xi_n) = -i2^{n/2-2} \frac{\langle 1i \rangle^2 \langle (i+1)n \rangle^2}{\langle 12 \rangle \dots \langle n1 \rangle} \left(1 + \frac{\langle 1(i+1) \rangle \langle in \rangle}{\langle 1i \rangle \langle (i+1)n \rangle} \right) \quad (3.4)$$

and an additional tower of vertices with a pair of scalars and an arbitrary number of positive helicity gluons that is generated from the transformation of the mass term:

$$V_{\text{CSW}}(\bar{\xi}_1, B_2, \dots, \xi_n) = i2^{n/2-1} \frac{-m^2 \langle 1n \rangle}{\langle 12 \rangle \dots \langle (n-1)n \rangle} \quad (3.5)$$

These vertices are connected by the usual massive scalar propagator

$$D_{\bar{\xi}\xi}(p^2) = \frac{i}{p^2 - m^2} \quad (3.6)$$

and scalar propagators i/p^2 connecting positive and negative-helicity gluons. Spinors corresponding to off-shell gluons and both on-shell and off-shell scalars are understood as usual in the CSW rules [5] and are obtained from the momentum by contraction with an arbitrary but fixed anti-holomorphic spinor η^a :

$$k_{\dot{\alpha}} = k_{\dot{\alpha}a} \eta^a \quad (3.7)$$

⁵For $SU(N)$ there are additional color suppressed terms that we don't display here.

When these quantities appear in spinor products we use the equivalent notations $\langle pk \rangle = \langle p - |k|\eta-\rangle = \langle p(k\eta) \rangle$.

The difference of the above rules with those in [6, 7] is that their formalism applies to a scalar space-time action which include a ϕ^4 space-time vertex. This coupling is required by the supersymmetry used there to derive the formalism. In contrast, our rules are those for a minimally coupled scalar.

The twistor-space structure of massive amplitudes resulting from the above rules was discussed in [37] where it was also verified that the amplitudes with a pair of massive scalars and only positive helicity gluons derived from the CSW rules satisfy the required on-shell recursion relation [41, 42]. The rules were also used to simplify the proof of the BCFW recursion relations for amplitudes with massive scalars [40]. Since the extra vertices are proportional to the mass, the holomorphic representation also provide a simple method to obtain the leading part of massive amplitudes in the small-mass limit [37].

One could also contemplate a transformation that transforms the mass term into a quadratic term of the ξ fields instead of the tower of vertices (3.5) and therefore removes the non-MHV type couplings completely. However, this apparently also removes the amplitudes with only positive helicity gluons. Since they are known to be non-vanishing, for such a transformation they are likely to be generated by the “equivalence theorem violation” mechanism [32] reviewed in section 2.5. Here we do not attempt to find such a transformation.

3.2 Tree level examples

As input for higher-point amplitudes, we note that inserting the prescription (3.7) and using momentum conservation, the cubic mass-vertex (3.5) can be written as

$$V(\bar{\xi}_1, B_2, \xi_3) = -\sqrt{2}im^2 \frac{\langle 13 \rangle}{\langle 12 \rangle \langle 23 \rangle} = \frac{\sqrt{2}im^2 [2\eta]}{\langle 2 - |k_3|\eta-\rangle} \quad (3.8)$$

where momentum conservation was used. This expression can be shown to be equivalent to the corresponding vertex resulting from the original action (3.1) provided all particles are on-shell [37]. In order to simplify calculations, it is useful to note that this vertex vanishes if the spinor $|\eta-\rangle$ is chosen in terms of the momentum of the gluon entering the vertex (this is not possible for the on-shell three particle amplitude [37]).

In [37] the CSW rules for massive scalars have been verified for three- and four-point scattering amplitudes. As a more stringent check, in this paper we consider some five point-functions. As shown in figure 2, there are four diagrams contributing to the five point function with only positive helicity gluons. Setting $|\eta-\rangle = |2-\rangle$ eliminates the last two diagrams that contain g_2^+ entering a three point vertex.

Using the prescription (3.7) for both the internal off-shell momenta and the external

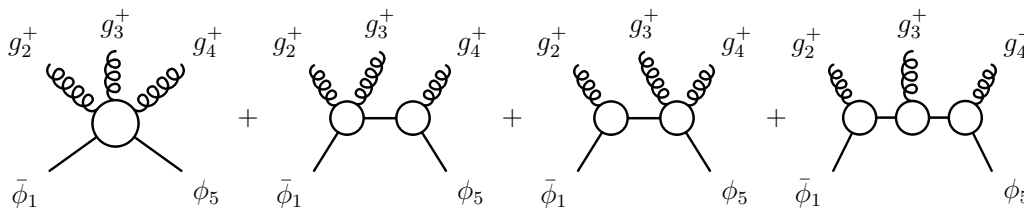


Figure 2: CSW-diagrams for the five-point amplitude with positive helicity gluons

scalars, the remaining two diagrams give (using the notation $y_{i,j} = k_{i,j}^2 - m^2$)

$$\begin{aligned}
 A_5(\bar{\xi}_1^+, B_2, B_3, B_4, \xi_5) &= \frac{-2^{3/2}im^2 \langle 15 \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle} + \frac{-2im^2 \langle 1k_{1,3} \rangle}{\langle 12 \rangle \langle 23 \rangle \langle 3k_{1,3} \rangle} \frac{i}{k_{1,3}^2 - m^2} \frac{-\sqrt{2}im^2 \langle k_{1,3,5} \rangle}{\langle k_{1,3,4} \rangle \langle 45 \rangle} \\
 &= \frac{-2^{3/2}im^2 (y_{1,3} \langle 2 + |k_1 k_5 | 2 - \rangle + m^2 \langle 2 + |k_3 k_4 | 2 - \rangle)}{y_{1,2} y_{1,3} \langle 23 \rangle \langle 34 \rangle \langle 4 - |k_5 | 2 - \rangle}
 \end{aligned} \tag{3.9}$$

This can be shown to be equivalent to the known result [53, 40, 41].

The structure of the amplitudes with one negative helicity gluon is more involved. Consider the amplitude $A_5(\bar{\xi}_1, \bar{B}_2, B_3, B_4, \xi_5)$. One class of diagrams contributing to the five point amplitude is similar to figure 2 but with the vertex coupled to \bar{B}_2 replaced by a massless MHV vertex. Setting $|\eta+\rangle = |4+\rangle$ kills the second and the last diagram. In addition there are two diagrams with an internal gluon line connecting the scalar mass-vertex with a gluonic MHV vertex (see figure 3) One gets

$$\begin{aligned}
 A_5(\bar{\xi}_1, \bar{B}_2, B_3, B_4, \xi_5) &= \frac{2^{3/2}i \langle 12 \rangle^2 \langle 25 \rangle^2}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 51 \rangle} + \frac{2^{3/2}im^2 \langle 12 \rangle \langle 2k_{1,2} \rangle \langle k_{1,2,5} \rangle}{\langle 1k_{1,2} \rangle (k_{1,2}^2 - m^2) \langle k_{1,2,3} \rangle \langle 34 \rangle \langle 45 \rangle} \\
 &+ \frac{2^{3/2}im^2 \langle 15 \rangle \langle (-k_{2,4}) 2 \rangle^3}{\langle 1k_{2,4} \rangle \langle k_{2,4,5} \rangle k_{2,4}^2 \langle 23 \rangle \langle 34 \rangle \langle 4(-k_{2,4}) \rangle} \\
 &+ \frac{2^{3/2}im^2 \langle 15 \rangle \langle (-k_{2,3}) 2 \rangle^3}{\langle 1k_{2,3} \rangle \langle k_{2,3,4} \rangle \langle 45 \rangle k_{2,3}^2 \langle 23 \rangle \langle 3(-k_{2,3}) \rangle}
 \end{aligned} \tag{3.10}$$

Making the choice $|\eta+\rangle = |4+\rangle$ explicit this becomes

$$\begin{aligned}
 A_5(\bar{\xi}_1, \bar{B}_2, B_3, B_4, \xi_5) &= \frac{2^{3/2}i \langle 4 + |k_1 | 2 + \rangle \langle 2 - |k_5 | 4 - \rangle^2}{y_{4,5} \langle 23 \rangle \langle 34 \rangle \langle 4 + |k_5 k_1 | 4 - \rangle} + \frac{2^{3/2}im^2 \langle 4 + |k_1 | 2 + \rangle [43]}{y_{1,2} y_{4,5} [24] \langle 34 \rangle} \\
 &- \frac{2^{3/2}im^2 [34]^3 \langle 23 \rangle^2}{\langle 4 + |k_1 k_5 | 4 - \rangle k_{2,4}^2 \langle 34 \rangle \langle 4 - |k_{2,3} | 4 - \rangle} \\
 &+ \frac{2^{3/2}im^2 [34]^3}{\langle 4 + |k_{2,3} | 4 + \rangle y_{4,5} [32] [42]}
 \end{aligned} \tag{3.11}$$

This expression agrees numerically with the known result [40, 41]. Although yielding less compact results than BCFW recursion relations, the calculation is still much simpler than the one using conventional Feynman rules, since no more combinatorics associated to the gluon-vertices has to be performed.

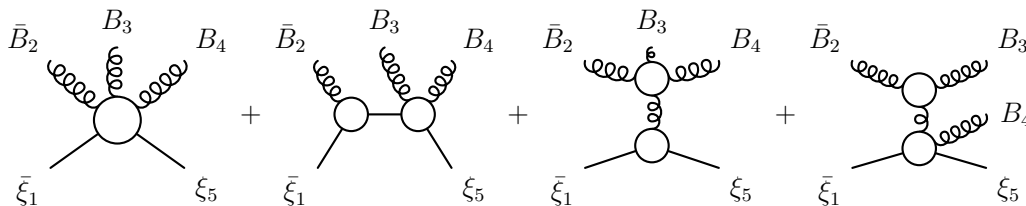


Figure 3: Non-vanishing CSW-diagrams for the five-point amplitude with one negative helicity gluon for the off-shell continuation $|\eta-\rangle = |4-\rangle$

3.3 Light-cone derivation of CSW rules for massive scalars

In this subsection we derive the CSW rules for massive scalars in the framework of canonical transformations in the light-cone gauge. We obtain the light-cone gauge Lagrangian for a scalar and derive the field redefinitions. Since, as emphasised before, the same field redefinition as for massless scalars is used, the structure of the transformation is similar to that for massless quarks proposed in [27].⁶ The explicit form of the expansion coefficients is related in a simple way to that for gluons [44] as suggested by supersymmetry [34]. These results are then used to derive the vertex (3.5) arising from the transformation of the mass term. The origin of the difference of the four-scalar vertex (3.4) to the results in the literature obtained from supersymmetry [6, 7] is also discussed.

Light-cone Yang-Mills with a massive scalar. In the first step of the derivation of the canonical transformation, the sum of the Yang-Mills Lagrangian and the Lagrangian of the massive scalars (3.1) is expressed entirely in terms of the physical gluon components ($A_z/A_{\bar{z}}$) and the scalars. An analogous discussion for the light-cone gauge Lagrangian of a scalar in the adjoint representation can be found in [29]. Imposing the light-cone gauge condition $A_+ = 0$ on the gluon, the scalar Lagrangian (3.1) can be expressed in terms of the surviving light-cone components A_- and $A_{z/\bar{z}}$. Making the group indices of the vector potential and the scalars explicit, the result is

$$\mathcal{L}_\phi = -\phi_i^\dagger(\square + m^2)\phi_i + ig \left[A_{-,ij}(\phi_i^\dagger \overleftrightarrow{\partial}_+ \phi_j) - A_{\perp,ij}(\phi_i^\dagger \overleftrightarrow{\partial}_\perp \phi_j) \right] + (ig)^2(\phi^\dagger A_\perp)_i(A_\perp \phi)_i \quad (3.12)$$

Here the abbreviation $\phi \overleftrightarrow{\partial} \psi = \phi(\partial\psi) - (\partial\phi)\psi$ and the scalar product of the transverse components $A_\perp B_\perp \equiv A_z B_{\bar{z}} + A_{\bar{z}} B_z$ have been introduced. As in the purely gluonic case, the component A_- appears only quadratically in the Lagrangian so it can be eliminated using the equation of motion. However, it now receives an additional contribution from the scalars:

$$A_- = \frac{1}{\partial_+^2} \left([D_\perp, \partial_+ A_\perp] + i\frac{g}{2}(\phi^\dagger \overleftrightarrow{\partial}_+ \phi) \right) \quad (3.13)$$

Inserting this back into the scalar Lagrangian (3.12) and the gluon Lagrangian, one obtains the interactions of the scalars with the physical modes of the gluons and a quartic self-coupling induced by the scalar contribution to A_- [29]:

$$\mathcal{L}_{\phi\phi}^{(2)} + \mathcal{L}_{\phi A_z \phi}^{(3)} + \mathcal{L}_{\phi A_{\bar{z}} \phi}^{(3)} + \mathcal{L}_{\phi A_z A_{\bar{z}} \phi}^{(4)} + \mathcal{L}_{\phi\phi\phi\phi}^{(4)} \quad (3.14)$$

⁶**Note added in proof:** The transformation for quarks has been worked out explicitly very recently in [54] which appeared while this article was in the final stages of preparation.

The cubic interactions can be written as

$$\mathcal{L}_{\phi A_z \phi}^{(3)} = 2ig \left[(\partial_{\bar{z}} \phi^\dagger) A_z \phi - (\partial_+ \phi^\dagger) \left(\frac{\partial_{\bar{z}}}{\partial_+} A_z \right) \phi \right] \quad (3.15)$$

$$\mathcal{L}_{\phi A_{\bar{z}} \phi}^{(3)} = 2ig \left[(\partial_z \phi^\dagger) A_{\bar{z}} \phi - (\partial_+ \phi^\dagger) \left(\frac{\partial_z}{\partial_+} A_{\bar{z}} \right) \phi \right] \quad (3.16)$$

Going to momentum space and translating to spinor notation one obtains the familiar vertices in the helicity formalism [40], for instance the coupling to a positive helicity gluon:

$$V(\phi_1^\dagger, A_{z,2}, \phi_3) = (2i)(\epsilon(k_2)^+)_z \frac{p_{1+} p_{2\bar{z}} - p_{1\bar{z}} p_{2+}}{p_{2+}} = \sqrt{2}i \frac{\langle 2 + | k_1 | \eta + \rangle}{\langle \eta 2 \rangle} \quad (3.17)$$

For completeness we also give the quartic interactions for a scalar in the fundamental representation

$$\mathcal{L}_{\phi A_z A_{\bar{z}} \phi}^{(4)} = (ig)^2 \left[(\phi^\dagger A_\perp)_i (A_\perp \phi)_i + ([\partial_+ A_\perp, A_\perp]_{ij}) (\partial_+)^{-2} (\phi_j^\dagger \overleftrightarrow{\partial}_+ \phi_i) \right] \quad (3.18)$$

$$\mathcal{L}_{\phi \phi \phi \phi}^{(4)} = \left(\frac{ig}{2} \right)^2 (\phi_i^\dagger \overleftrightarrow{\partial}_+ \phi_j) (\partial_+)^{-2} (\phi_j^\dagger \overleftrightarrow{\partial}_+ \phi_i) \quad (3.19)$$

Field redefinition for scalars. In the second step, the non-MHV vertices contained in the gluon and the scalar Lagrangian are eliminated by an appropriate canonical transformation. As in the case of the pure Yang-Mills Lagrangian reviewed in section 2.2, a canonical transformation to new gluon variables B and momenta $\partial_+ \bar{B}$ can be used in order to eliminate the cubic gluonic non-MHV coupling. In addition, there is now the non-MHV type scalar interaction $\mathcal{L}_{\phi A_z \phi}^{(3)}$ that can be eliminated by transforming also to new scalar field variables and momenta ξ and $\Pi_\xi \sim \partial_+ \bar{\xi}$.

Following closely the proposal for the incorporation of massless quarks in [27], the transformation of the positive helicity gluon is taken independent of the scalars so that the generating functional of the transformation is of the form

$$G[A_z, \phi, \Pi_B, \Pi_\xi] = \int d^3 y \left(B[A_z(\vec{y})] \Pi_B(\vec{y}) + \xi[\phi(\vec{y}), A_z(\vec{y})] \Pi_\xi(\vec{y}) \right) \quad (3.20)$$

A common dependence on the fixed 'time' x^- has been suppressed. The transformation of the scalars are induced via the relations $\xi = \delta G / \delta \Pi_\xi$ and $\Pi_\phi = \delta G / \delta \phi$:

$$\phi \rightarrow \xi[\phi, A_z] \quad \Pi_\phi \sim \partial_+ \phi^\dagger(x^-, \vec{x}) = \int d^3 y \partial_+ \bar{\xi}(x^-, \vec{y}) \frac{\delta \xi(x^-, \vec{y})}{\delta \phi(x^-, \vec{x})} \quad (3.21)$$

Since ξ depends on the A_z there is also a contribution from the scalars to the canonical momentum $\partial_+ A_{\bar{z}} \sim \delta G / \delta A_z$, in addition to (2.28):

$$\partial_+ A_{\bar{z}}(x^-, \vec{x}) = \int d^3 y \left(\frac{\delta B(x^-, \vec{y})}{\delta A_z(x^-, \vec{x})} \partial_+ \bar{B}(x^-, \vec{y}) + \frac{1}{2} \partial_+ \bar{\xi}(x^-, \vec{y}) \frac{\delta \xi(x^-, \vec{y})}{\delta A_z(x^-, \vec{x})} \right) \quad (3.22)$$

The second term ensures that the sum of the kinetic terms is invariant, i.e.

$$\int d^3 x (2\partial_+ \bar{B} \partial_- B + \partial_+ \bar{\xi} \partial_- \xi) = \int d^3 x (2\partial_+ A_{\bar{z}} \partial_- A_z + \partial_+ \phi^\dagger \partial_- \phi) \quad (3.23)$$

since ξ depends on x^- implicitly through both ϕ and A_z so that

$$\partial_- \xi(x^-, \vec{x}) = \int d^3 y \left[\frac{\delta \xi(\vec{x})}{\delta \phi(\vec{y})} \partial_- \phi(x^-, \vec{y}) + \frac{\delta \xi(\vec{x})}{\delta A_z(\vec{y})} \partial_- A_z(x^-, \vec{y}) \right] \quad (3.24)$$

The ξ -dependent contribution to the conjugate gluon-momentum $\partial_+ A_z$ will result in an additional term in the condition of the gluonic transformation (2.29) so the transformations of the scalars and the gluons cannot be treated separately. Therefore the condition to eliminate the non-MHV type vertices is

$$\int d^3 x \left[4A_z \partial_z \partial_{\bar{z}} A_z + 2\phi^\dagger \partial_z \partial_{\bar{z}} \phi + \mathcal{L}_{++-}^{(3)} + \mathcal{L}_{\phi A_z \phi}^{(3)} \right] = \int d^3 x \left[4\bar{B} \partial_z \partial_{\bar{z}} B + 2\bar{\xi} \partial_z \partial_{\bar{z}} \xi \right] \quad (3.25)$$

Since the mass term of the scalar has not been included in this definition, the transformation has the same form for massive and massless scalars. Consequently the mass term $m^2 \phi^\dagger \phi$ will not be invariant but instead be transformed into a tower of vertices $m^2 \bar{\xi} B \dots B \xi$ resulting in the CSW vertices (3.5).

The conditions (3.25) and (3.21) can be solved by methods similar to the ones used in pure Yang-Mills theory in [44]. The transformation of the positive helicity gluons $A_z[B]$ will be taken identical to the pure Yang-Mills case defined by (2.29). Inserting the expressions for the canonical momenta (3.21) and (3.22) into (3.25) the vanishing of the coefficient of $\partial_+ \bar{\xi}(\vec{x})$ leads to the condition

$$\begin{aligned} \frac{\partial_z \partial_{\bar{z}}}{\partial_+} \xi(\vec{x}) + \int d^3 y d^3 w \operatorname{tr} \left[\left(\frac{\partial_z \partial_{\bar{z}}}{\partial_+} B(\vec{w}) \right) \frac{\delta \xi(\vec{x})}{\delta \phi(\vec{y})} \frac{\delta \phi(\vec{y})}{\delta B(\vec{w})} \right] \\ = \int d^3 y \left[\frac{\partial_z \partial_{\bar{z}}}{\partial_+} \phi(\vec{y}) + ig \left(\frac{\partial_z}{\partial_+} A_z(\vec{y}) \right) \phi(\vec{y}) - ig \frac{\partial_z}{\partial_+} \left(A_z(\vec{y}) \phi(\vec{y}) \right) \right] \frac{\delta \xi(\vec{x})}{\delta \phi(\vec{y})} \end{aligned} \quad (3.26)$$

The second term on the left-hand side arises from inserting the scalar contribution to the canonical momentum (3.22) into the cubic gluon vertex $\mathcal{L}_{++-}^{(3)}$ and the gluon kinetic term. To arrive at this form it has been used that $A_z[B]$ satisfies the same relation (2.29) as in the purely gluonic case. This implies the relation

$$\begin{aligned} \int d^3 y \left(\frac{\partial_z \partial_{\bar{z}}}{\partial_+} A_z(\vec{y}) - ig \left[A_z(\vec{y}), \frac{\partial_z}{\partial_+} A_z(\vec{y}) \right] \right) \frac{\delta \xi(\vec{x})}{\delta A_z(\vec{y})} \\ = - \int d^3 y d^3 z \left(\frac{\partial_z \partial_{\bar{z}}}{\partial_+} B(\vec{z}) \right) \frac{\delta \xi(\vec{x})}{\delta \phi(\vec{y})} \frac{\delta \phi(\vec{y})}{\delta B(\vec{z})} \end{aligned} \quad (3.27)$$

where it was used that the new coordinates $B[A_z]$ and $\xi[\phi, A_z]$ are independent variables so that $0 = \delta \xi_x / \delta B_z = \int d^3 y (\delta \xi_x / \delta \phi_y) (\delta \phi_y / \delta B_z) + (\delta \xi_x / \delta A_y) (\delta A_y / \delta B_z)$.

Following [44] a recursion relation can be derived from the condition (3.26) by multiplying with $\delta \phi(\vec{z}) / \delta \xi(\vec{x})$, integrating over \vec{x} and transforming to momentum space:

$$\int \widetilde{d^3 p} \left(\frac{\delta \phi_{k_1}}{\delta \xi_p} \omega_p \xi_p + \frac{\delta \phi_{k_1}}{\delta B_p} \omega_p B_p \right) = \omega_{k_1} \phi_{k_1} + g \int \widetilde{d^3 k_2} (\zeta_{k_1} - \zeta_{k_2}) A_{k_2, z} \phi_{k_1 - k_2} \quad (3.28)$$

Note that here the relations $\xi[\phi, A_z]$ and $B[A_z]$ have been inverted resulting in the old fields as function of the new ones, $A_z[B]$ and $\phi[\xi, B]$.

To solve the condition (3.28) we make an Ansatz for $\phi[\xi, B]$ that is linear in ξ :

$$\phi(p) = \sum_{n=1}^{\infty} \int \prod_{i=1}^n \widetilde{dk}_i \mathcal{Z}(p, k_1, \dots, k_n) B_{-k_1} \dots B_{-k_{n-1}} \xi_{-k_n} \quad (3.29)$$

Inserting this together with the solution for A_z (2.30) into the condition (3.28) gives rise to a recursion relation for the coefficients \mathcal{Z} :

$$\omega_{k_1, n} \mathcal{Z}(k_1, \dots, k_n) = g \sum_{i=2}^{n-1} (\zeta_{k_2, i} - \zeta_{k_1}) \mathcal{Y}(-k_{2, i}, k_2, \dots, k_i) \mathcal{Z}(-k_{i+1, n}, k_{i+1}, \dots, k_n) \quad (3.30)$$

Analogously to the gluon case, the factor multiplying the expansion coefficients is proportional to the vertex (3.17) that we are trying to eliminate. In fact, using momentum conservation it is easily seen that the factor is proportional to that in the gluonic case (2.32):

$$\zeta_{k_2, i} - \zeta_{k_1} = -\frac{k_{(2, i)\bar{z}} k_{(i+1, n)+} - k_{(2, i)+} k_{(i+1, n)\bar{z}}}{k_{(2, i)+} k_{1+}} = \frac{k_{(i+1, n)+}}{k_{1+}} (\zeta_{k_{i+1, n}} - \zeta_{k_2, i}) \quad (3.31)$$

Therefore the solution to the recursion relation (3.30) is simply proportional to the gluonic expansion coefficients \mathcal{Y} . The solution satisfying the normalisation condition $\mathcal{Z}(p, -p) = 1$ is given by

$$\mathcal{Z}(p, k_1, \dots, k_n) = -\frac{k_{n+}}{p_+} \mathcal{Y}(p, k_1, \dots, k_n) = (g\sqrt{2})^{n-1} \frac{\langle \eta n \rangle}{\langle \eta 1 \rangle \langle 12 \rangle \dots \langle (n-1)n \rangle} \quad (3.32)$$

This expansion can also be inferred from the proposed transformation of the light-cone $\mathcal{N} = 4$ SUSY superfield [34]. The fact that all six real scalars of the $\mathcal{N} = 4$ SUSY-multiplet have the same expansion [34] suggests that the expansion coefficients for the conjugate momentum ϕ_p^\dagger are the same as those for ϕ :

$$\phi_p^\dagger = \sum_{n=1}^{\infty} \int \prod_{i=1}^n \widetilde{dk}_i (g\sqrt{2})^{n-1} \frac{\langle \eta 1 \rangle \bar{\xi}_{-k_1} B_{-k_2} \dots B_{-k_n}}{\langle \eta n \rangle \langle 12 \rangle \dots \langle (n-1)n \rangle} \quad (3.33)$$

In order to verify that this is the correct expansion one would have to insert the inverse of the transformation $\phi \rightarrow \xi[\phi, A_z]$ into the defining equation of the canonical momentum of the scalars (3.21). In the same way one would also obtain the coefficients that enter the expansion of the additional scalar contribution to the momentum of the gluons in (3.22) in terms of the B -fields. Instead of attempting to invert the transformation, here we follow the path used in [44] where the transformation for the conjugate momentum $\partial_+ A_{\bar{z}}$ was constructed by demanding that the kinetic term for the gluons remains invariant. For the case at hand one can only demand that the *sum* of the kinetic terms of scalars and gluons remains invariant:

$$(p_+ \phi_p^\dagger)(p_- \phi_{-p}) + 2 \operatorname{tr}[(p_+ A_{p, \bar{z}})(p_- A_{-p, z})] = (p_+ \bar{\xi}_p)(p_- \xi_{-p}) + 2 \operatorname{tr}[(p_+ \bar{B}_p)(p_- B_{-p})] \quad (3.34)$$

As demonstrated in appendix C, this requirement together with the expansions (3.29) and (3.33) fixes the expansion of the gluon momentum to be

$$p_+ A_{p, \bar{z}} = \sum_{n=1}^{\infty} \int \sum_{s=1}^n \prod_{i=1}^n \widetilde{dk}_i \frac{-(g\sqrt{2})^{n-1} \langle \eta s \rangle^2}{\langle \eta 1 \rangle \langle 12 \rangle \dots \langle (n-1)n \rangle \langle n\eta \rangle} B_{-k_1} \dots B_{-k_{s-1}} \left((k_s)_+ \bar{B}_{-k_s} B_{-k_{s+1}} + \xi_{-k_s} (k_{s+1})_+ \bar{\xi}_{-k_{s+1}} \right) B_{-k_{s+1}} \dots B_{-k_n} \quad (3.35)$$

This completes the construction of the canonical transformation for a scalar in the fundamental representation. The results obtained so far apply equally well to massless and massive scalars.

Derivation of the vertices. The expansion of the scalars (3.29) and (3.33) can now be used to determine the additional tower of vertices generated by the transformation of the mass term $m^2\phi^\dagger\phi$:

$$-m^2\phi_p^\dagger\phi_{-p} = \sum_{n=2}^{\infty} \int \prod_{i=1}^n \widetilde{dk}_i \mathcal{V}_{\bar{\xi}_1, B_2, \dots, B_{n-1}, \xi_n}(\bar{\xi}_{k_1} B_{k_2} \dots B_{k_{n-1}} \xi_{k_n}) \quad (3.36)$$

where the vertex-function can be simplified using the eikonal identity (B.1):

$$\mathcal{V}_{\bar{\xi}_1, B_2, \dots, B_{n-1}, \xi_n} = \sum_{j=1}^{n-1} \frac{-m^2(g\sqrt{2})^{n-2} \langle \eta 1 \rangle \langle \eta n \rangle}{\langle 12 \rangle \dots \langle \eta j \rangle \langle \eta(j+1) \rangle \dots \langle (n-1)n \rangle} = \frac{-m^2(g\sqrt{2})^{n-2} \langle 1n \rangle}{\langle 12 \rangle \dots \langle (n-1)n \rangle} \quad (3.37)$$

For the case $n = 2$ we just obtain a mass-term $-m^2\bar{\xi}\xi$ that can be re-summed into the scalar propagator. Since in the convention for colour-ordering (A.3) a factor g^{n-2} is stripped off, the colour ordered vertex generated by the mass term is given by (3.5).

This establishes the main new result of the rules given in section 3.1. The other vertices are not specific to massive scalars but we briefly comment on their origin in the light-cone formalism. The four-scalar vertex (3.4) differs from the result for $\mathcal{N} = 4$ supersymmetric theories

$$\mathcal{V}_{\bar{\xi}_1, B_2, \dots, \xi_j, \bar{\xi}_{j+1}, \dots, \xi_n}^{\mathcal{N}=4} = -(g\sqrt{2})^{n-2} \frac{\langle 1j \rangle^2 \langle (j+1)n \rangle^2}{\langle 12 \rangle \dots \langle n1 \rangle} \quad (3.38)$$

considered previously in the literature [6–8] since the $\mathcal{N} = 4$ SUSY action contains a quartic scalar coupling, in contrast to the action (3.1). The transformation of a ϕ^4 interaction is obtained analogously to that of the mass term:

$$\lambda(\phi^\dagger\phi)^2 = \sum_{n=4}^{\infty} \sum_{j=2}^{n-1} \int \prod_{i=1}^n \widetilde{dk}_i \mathcal{V}_{\bar{\xi}_1, B_2, \dots, \xi_j, \bar{\xi}_{j+1}, \dots, \xi_n}^\lambda(\bar{\xi}_{k_1} B_{k_2} \dots \xi_{k_j})(\bar{\xi}_{k_{j+1}} \dots B_{k_{n-1}} \xi_{k_n}) \quad (3.39)$$

with the vertex function

$$\mathcal{V}_{\bar{\xi}_1, B_2, \dots, \xi_j, \bar{\xi}_{j+1}, \dots, \xi_n}^{\phi^4} = \lambda(g\sqrt{2})^{n-4} \frac{\langle 1j \rangle \langle j(j+1) \rangle \langle (j+1)n \rangle}{\langle 12 \rangle \dots \langle (n-1)n \rangle} \quad (3.40)$$

Subtracting this expression with $\lambda = -g^2$ from the SUSY vertex (3.39) leads precisely to the vertex (3.4) appropriate for the Lagrangian (3.1). Of course it is also straightforward to obtain the CSW-rules for a scalar with a genuine ϕ^4 coupling from these result.

To complete the proof of the CSW rules for scalars in the light-cone formalism would require to show explicitly that the remaining interaction vertices are transformed into towers of the massless MHV vertices (3.3) and (3.4). Since we use the same transformation for massive scalars as for massless ones, this result is expected. However, the explicit derivation is more complicated than in the case of pure Yang-Mills — where it was done only for the four and five-point vertices [44] — because of the the additional scalar contributions

in the transformation of $\partial_+ A_{\bar{z}}$ in (3.22). Therefore the scalar MHV vertex (3.3) receives contributions from four sources: the transformed scalar-gluon vertices $\mathcal{L}_{\phi^\dagger A_{\bar{z}} \phi}^{(3)}$ and $\mathcal{L}_{\phi^\dagger A_z A_{\bar{z}} \phi}^{(4)}$ but also from the cubic and quartic gluon vertices $\mathcal{L}_{+++}^{(3)}$ and $\mathcal{L}_{+++-}^{(4)}$. Similarly, the vertices $\mathcal{L}_{\phi^\dagger A_{\bar{z}} \phi}^{(3)}$, $\mathcal{L}_{\phi^\dagger A_z A_{\bar{z}} \phi}^{(4)}$, $\mathcal{L}_{\phi^\dagger \phi \phi^\dagger \phi}^{(4)}$ and $\mathcal{L}_{++--}^{(4)}$ contribute to the four-scalar MHV vertex (3.4). Given the fact that the form of the vertices (3.3) and (3.4) is not specific to massive scalars, which are our main concern in this paper, and since there is no reason to suspect that the proposal of [27] should fail for massless scalars we do not attempt to derive the MHV vertices here, but the transformations (3.29) and (3.35) provide all the necessary ingredients to do so.

3.4 Twistor-Yang Mills with massive scalars

The twistor action for a scalar coupled to Yang-Mills theory can be derived through the Noether procedure. This is done in full detail in appendix D. It is natural to lift a scalar field and its complex conjugate separately to twistor space. This is because a first-order action is preferred.⁷ A simpler way to see this is to study the scalar couplings in the $\mathcal{N} = 4$ twistor action. This fixes the twistor lift of the covariant kinetic terms in the space-time action. For the mass term we need the explicit lifting formula for the scalar (charged under the fundamental representation of the gauge group),

$$\phi(x) = \int_{\mathbb{CP}^1} H^{-1} \xi_0(x, \pi) \quad (3.41)$$

$$\phi^\dagger(x) = \int_{\mathbb{CP}^1} \bar{\xi}_0(x, \pi) H \quad (3.42)$$

so the mass term becomes

$$\mathcal{L}_{\text{mass}} = -m^2 \text{tr} \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} (\bar{\xi}_0(x, \pi_1) H)_1 (H^{-1} \xi_0(x, \pi_2))_2 \quad (3.43)$$

Putting everything together the action 3.1 lifts to

$$S[\xi, \bar{\xi}, B, \bar{B}] = S_{\text{YM}} + 2 \frac{1}{2} \int \Omega \wedge (\bar{\xi} \wedge (\bar{\partial} \xi - i\sqrt{2}g[B, \xi])) + S_{\bar{B}, \xi, \bar{\xi}} + S_{\xi^2(\bar{\xi})^2} + S_{\text{mass}} \quad (3.44)$$

with

$$S_{\bar{B}, \xi, \bar{\xi}} = 2 \frac{1}{2} \sqrt{2}g \int_{\mathbb{R}^4} d^4x \int_{(\mathbb{CP}^1)^3} \frac{\langle \pi_1 \pi_2 \rangle \langle \pi_2 \pi_3 \rangle}{\langle \pi_1 \pi_3 \rangle} (\bar{\xi}_1^0 H_1 H_2^{-1} \bar{B}_2^0 H_2 H_3^{-1} \xi_3^0) \prod_{i=1}^3 D\pi_i \quad (3.45)$$

$$S_{\xi^2(\bar{\xi})^2} = g^2 \int_{\mathbb{R}^4} d^4x \int_{(\mathbb{CP}^1)^4} \left(2 \frac{\langle \pi_1 \pi_2 \rangle \langle \pi_3 \pi_4 \rangle}{\langle \pi_2 \pi_3 \rangle \langle \pi_4 \pi_1 \rangle} - 1 \right) \frac{1}{2} (\bar{\xi}_1^0 H_1 H_2^{-1} \xi_2^0 \bar{\xi}_3^0 H_3 H_4^{-1} \xi_4^0) \quad (3.46)$$

as an action on twistor space. In addition to the gauge symmetries for the Yang-Mills part, the full action is invariant under

$$\xi \rightarrow \xi + \bar{\partial}_B f^{-2} \quad \bar{B} \rightarrow \bar{B} - i\sqrt{2}g[\bar{\xi}, f^{-2}] \quad (3.47)$$

⁷As stated before basic twistor theory relates weighted Dolbeault cohomology classes with solutions to the wave equation. The closedness condition is a first order differential equation.

and

$$\bar{\xi} \rightarrow \bar{\xi} + \bar{\partial}_B \bar{f}^{-2} \quad \bar{B} \rightarrow \bar{B} - i\sqrt{2}g[\xi, \bar{f}^{-2}] \quad (3.48)$$

for twistor gauge parameters of the indicated weight. As before, this symmetry must be fixed in order to derive scattering rules (or more precisely to invert the kinetic operator).

Gauge choices. A space-time gauge can be imposed on all the twistor fields, including the scalars. The action 3.44 reduces in that case directly to 3.1. This can be derived straightforwardly from [33], up to the subtlety mentioned below. There is in this case no remaining residual gauge symmetry for the scalar fields.

This gauge choice explains the ‘-1’ in the vertex for the four point interaction as this factor is necessary to avoid a ϕ^4 term in the action. In space-time gauge, without the ‘-1’, a ϕ^4 interaction vertex would arise, which the added factor cancels. To see this, note that in space-time gauge

$$\bar{\partial}_0^\dagger \xi_0 = 0 \quad \rightarrow \quad \xi_0 = \phi(x) \quad (3.49)$$

holds. Therefore only the remaining integral over the π ’s must be performed. Now it is convenient to recall the standard integrals

$$\int_{\mathbb{CP}^1} \frac{\pi^{\dot{\alpha}} \hat{\pi}^{\dot{\beta}}}{\langle \pi \hat{\pi} \rangle} = \frac{1}{2} \epsilon^{\dot{\alpha}\dot{\beta}} \quad (3.50)$$

$$\int_{\mathbb{CP}^1} \frac{\langle \pi_1 \xi \rangle}{\langle \pi_1 \nu \rangle} = \frac{\langle \xi \hat{\nu} \rangle}{\langle \nu \hat{\nu} \rangle} \quad (3.51)$$

for arbitrary spinors ξ and ν . Therefore

$$\int_{(\mathbb{CP}^1)^4} \frac{\langle \pi_1 \pi_2 \rangle \langle \pi_3 \pi_4 \rangle}{\langle \pi_2 \pi_3 \rangle \langle \pi_4 \pi_1 \rangle} = \frac{1}{2} \quad (3.52)$$

and the extra ‘-1’ is required to obtain the space-time action we are interested in. The underlying reason this arises is that the action lifted to twistor space is not the minimally coupled one. Interestingly, the ϕ^4 interaction is required by supersymmetry and, of course, is part of the Higgs potential and arises naturally on twistor space.

The second obvious gauge choice is the CSW-gauge,

$$\eta^\alpha (B_\alpha, \xi_\alpha, \bar{\xi}_\alpha, \bar{B}_\alpha) = 0 \quad (3.53)$$

which eliminates the interaction vertices in part of the Lagrangian. In this gauge the scalar fields have, without the mass term, the same interesting propagator as the gluon fields,

$$: \xi_0 \bar{\xi}_0 := i \frac{\delta(\eta \pi_1 p) \delta(\eta \pi_2 p)}{p^2} \quad (3.54)$$

Although this can be verified by direct computation, it is an expression of an effective supersymmetry since the $\mathcal{N} = 4$ action has the same kinetic term for all component fields in the local part of the action.⁸ The mass term changes things slightly: for the free

⁸The non-local part is different but can be shown not to contribute to this particular Green’s function

propagator the quadratic part has to be split from the other terms in this vertex. It can be verified that the correct propagator now simply is

$$: \xi_0 \bar{\xi}_0 := i \frac{\delta(\eta\pi_1 p) \delta(\eta\pi_2 p)}{p^2 - m^2} \quad (3.55)$$

which can be derived by the usual geometric series argument or direct computation. All this does not change much for the EM coefficients, which simply follow directly from the lifting formula as

$$\phi(x) = \int_{\mathbb{CP}^1} \frac{H^{-1}(\eta) H(\pi)}{\langle \eta \pi \rangle} \xi_0(x, \pi) \langle \eta \pi \rangle \quad (3.56)$$

Expanding the two dimensional propagator, plugging in the delta functions and Fourier transforming gives

$$\phi(p) = \sum_i \sum_{j=1}^i \int \left(\prod_{k=i}^j dp_k^4 \right) B_0(-p_1) \dots \xi_0(-p_j) \mathcal{Y}(p, p_1, \dots, p_j) \quad (3.57)$$

with

$$\mathcal{Y}(p, p_1, \dots, p_j) = (i\sqrt{2}g)^j \frac{\langle \eta \eta p_j \rangle}{\langle \eta(\eta p_1) \rangle \langle (\eta p_1)(\eta p_2) \rangle \dots \langle (\eta p_{j-1})(\eta p_j) \rangle} \quad (3.58)$$

Using the spinor momentum trick (2.8) and absorbing the factors $-i[\eta i]^2$ factors into the B_0 fields as in (2.59) this turns into

$$\mathcal{Y}(p, p_1, \dots, p_j) = (\sqrt{2}g)^j \frac{\langle \eta j \rangle}{\langle \eta 1 \rangle \langle 12 \rangle \dots \langle (j-1)j \rangle} \quad (3.59)$$

It is obvious that an identical result follows for ϕ^\dagger . This is of course nothing but (3.32) which was obtained through the canonical transformation.

Actually, it is interesting to note that the space-time interpretation of the transformation rule for one of the space-time helicities now gets an extra contribution compared to (2.61). This is in line with the extra contributions in (3.22) in the canonical approach. The reason here is that the \bar{B}_0 field equation used in deriving that equation gets an extra contribution from the $\bar{B}_0 \xi_0 \bar{\xi}_0$ vertex. Therefore we obtain the modified equation

$$(\eta \eta p) A_{\bar{z}} = -\frac{1}{\sqrt{2}} (\hat{\eta}^\alpha \eta_\alpha) \int_{\pi_k} dk \frac{H(\eta) H^{-1}(\pi_k)}{\langle \eta \pi_k \rangle} \bar{B}_0(\pi_k) (\langle \pi_k \eta \rangle)^4 \frac{H(\pi_k) H^{-1}(\eta)}{\langle \pi_k \eta \rangle} + \quad (3.60)$$

$$\int_{\pi_1, \pi_2} \frac{\langle \eta \pi_1 \rangle \langle \eta \pi_2 \rangle}{\langle \pi_1 \pi_2 \rangle} (H^{-1}(\eta) H \xi)_1 (\bar{\xi} H^{-1} H(\eta))_2 \quad (3.61)$$

Performing the by now familiar (we hope!) operations of inserting the polarisation vector and using the spinor momentum trick, we obtain

$$[\eta p]^2 \bar{A}(p) = \sum_{i,n} \frac{\langle \eta p_i \rangle^2 (\sqrt{2}g)^{n-1} B_0(p_1) \dots B_0(p_{i-1}) X_i B_0(p_{i+2}) \dots B_0(p_n)}{\langle \eta 1 \rangle \langle 23 \rangle \dots \langle (n-1)\eta \rangle} \quad (3.62)$$

with

$$X_i = \langle \eta p_i \rangle^2 \bar{B}(p_i) B(p_{i+1}) + \langle \eta p_{i+1} \rangle^2 \xi(p_i) \bar{\xi}(p_{i+1}) \quad (3.63)$$

in terms of the spinor products. Again, using (2.7) this is easily seen to correspond to (3.22). In these equations we have for clarity suppressed the field normalisation constants ($\sim i[\eta 1]^2$) and the color factors which are easily reinstated if one bears in mind that the scalars transform in the fundamental.

3.5 Equivalence theorem for massive scalars

In massless theories the non MHV-type cubic vertices that appear to be missing in the CSW Lagrangian are generated by violations of the naive 'equivalence theorem' as discussed in 2.5. For massive scalars the situation is different since the transformation of the mass term generates a non-MHV three-point vertex $V(\bar{\xi}, B, \xi)$ that is equivalent to the conventional vertex provided all particles are on-shell [37]. Here we discuss in more detail why 'equivalence theorem violating' contributions to this vertex are absent for massive scalars.

In the light-cone gauge the scalar non-MHV vertex arises from the Green's function $\langle 0 | \bar{\phi} A_z \phi | 0 \rangle$. Similar to the discussion of the cubic non-MHV gluon vertex (2.67) this Green's function is expressed as a function of the new fields (B, \bar{B}) and $(\xi, \bar{\xi})$ by inserting the transformations (3.29) and (3.35) or equivalently the relations to the twistor-fields (3.56) and (3.60):

$$\begin{aligned} \langle 0 | \bar{\phi}_{p_1} A_{p_2, \bar{z}} \phi_{p_3} | 0 \rangle &= \left\langle 0 \left| \bar{\xi}_{p_1} \bar{B}_{p_2} \xi_{p_3} + \int \widetilde{dk_1} \widetilde{dk_2} \left[(\mathcal{Z}(p_1, k_2, k_1) \bar{\xi}_{-k_1} B_{-k_2}) \bar{B}_{p_2} \xi_{p_3} \right. \right. \right. \\ &+ \left. \left. \bar{\xi}_{p_1} \left(\frac{k_{2+}}{p_{2+}} \mathcal{W}^1(p_2, k_2, k_1) \xi_{-k_1} \bar{\xi}_{-k_2} \right) \xi_{p_3} + \bar{\xi}_{p_1} \bar{B}_{p_2} (\mathcal{Z}(p_3, k_1, k_2) B_{-k_1} \xi_{-k_2}) \right] + \dots \right| 0 \rangle \end{aligned} \quad (3.64)$$

For the definition of the \mathcal{W} coefficients see (C.3). The on-shell scattering amplitude is obtained as usual by multiplying by $(p_1^2 - m^2)p_2^2(p_3^2 - m^2)$ and taking the on-shell limit.

For a massive scalar the first term is non-vanishing due to the vertex (3.5). Performing again the LSZ reduction on legs 1 and 2 first, the only additional contribution potentially comes from the last term:

$$\mathcal{Z}(p_3, p_2, p_1) = g\sqrt{2} \frac{\langle \eta 1 \rangle}{\langle \eta 2 \rangle \langle 21 \rangle} = g\sqrt{2} \frac{1}{\langle 12 \rangle [21]} \frac{\langle \eta - |k_1| 2 - \rangle}{\langle \eta 2 \rangle} \quad (3.65)$$

For a massless scalar with $p_1^2 = 0$ this gives the correct vertex $V(\bar{\phi}, A_z, \phi)$ times a propagator $1/(p_1 + p_2)^2$. For a massive scalar with $p_1^2 = m^2$ it can be seen using the identity (2.69) that the denominator contains instead

$$\langle 12 \rangle [21] = p_3^2 + m^2 \frac{(p_3 \cdot \eta)}{(p_1 \cdot \eta)} \quad (3.66)$$

so there is no pole for $p_3^2 \rightarrow m^2$ and the term vanishes upon LSZ reduction.

It is also instructive to see that there is no contribution from the on-shell limit of the gluon leg. Performing the reduction first on legs 1 and 3, the contribution from leg 2 gives

$$\frac{(p_3)_+}{(p_2)_+} \mathcal{W}^1(p_2, p_3, p_1) = g\sqrt{2} \frac{(p_3)_+}{(p_2)_+} \frac{\langle \eta 1 \rangle}{\langle \eta 3 \rangle \langle 13 \rangle} = g\sqrt{2} \frac{1}{[31] \langle 13 \rangle} \frac{\langle \eta 1 \rangle [12]}{\langle \eta 2 \rangle} \quad (3.67)$$

Applying (2.69) the denominator can be written as

$$\langle 13 \rangle [31] = p_2^2 - m^2 \frac{(\eta \cdot p_2)^2}{(p_1 \cdot \eta)(p_3 \cdot \eta)} \quad (3.68)$$

so there is no pole for $p_2^2 = 0$ in the massive case and again there is no 'equivalence theorem violating' contribution. Hence we have demonstrated explicitly that only for mass-less particles equivalence theorem evasion is expected to play a role for the field transformation employed in this article.

4. One-loop amplitudes in pure Yang-Mills

It was shown in [16–20] that the so called cut constructable pieces of one-loop amplitudes in pure Yang-Mills can be computed using the CSW rules. For some while the so called rational terms, however, remained elusive from the CSW point of view although several alternative methods are available to compute these (see e.g. [55] and references therein). By now several quantum completions of the CSW formalism in pure Yang-Mills have been proposed in the literature that also reproduce at least some of the rational parts of one-loop amplitudes. These employ

- ordinary lightcone loops [30]
- a Lorentz-violating regularisation [31]
- equivalence theorem violations [32]

all of which are slightly unsatisfactory for various reasons. The first method does not offer much advantages over ordinary lightcone methods. The second method gives a clear argument that the all-plus loop can be interpreted as an anomaly, but the regularisation used makes it hard in the general case to compare results obtained in it to ordinary dimensional regularisation for instance. Equivalence theorem violations show where both tree and loop amplitudes are hiding, but the formalism in D dimensions employed breaks supersymmetry explicitly. This obfuscates the use of supersymmetric decomposition for instance.

In this section we show that the CSW rules for massive scalars can be useful to address the issue of one-loop amplitudes in pure Yang-Mills within the context of conventional dimensional regularisation, in particular in the four-dimensional helicity scheme variant thereof. This is suggested by the decomposition of one-loop amplitudes in pure Yang-Mills theory according to the particles which run in the loop [56]

$$A_{\text{pure YM}} = A_{\mathcal{N}=4} - 4A_{\mathcal{N}=1} + A_{\text{scalar}} \quad (4.1)$$

The supersymmetric contributions are cut-constructable, so can be calculated using four dimensional unitarity or the one-loop MHV rules. The scalar part is much harder since it involves, from the four dimensional point of view, massive scalars. This can be seen by decomposing the $D = 4 - 2\epsilon$ dimensional momentum as $\ell_D = (\ell, \ell_{-2\epsilon})$ and taking the four-dimensional part and the -2ϵ dimensional part as orthogonal so that

$$\ell_D^2 = \ell^2 + \ell_{-2\epsilon}^2 \equiv \ell^2 - \mu^2 \quad (4.2)$$

and decomposing the dimensionally regulated loop integral measure as [56]

$$\frac{d^D \ell}{(2\pi)^D} = \frac{d^4 \ell}{(2\pi)^4} \frac{d^{-2\epsilon} \mu}{(2\pi)^{-2\epsilon}} \quad (4.3)$$

This suggests that our CSW rules for massive scalars given in section 3.1 should be suitable for the calculation of the rational parts of loop amplitudes. We now give several arguments that this is indeed the case and verify in section 4.1 that the one-loop four-point amplitude with positive helicity gluons is correctly reproduced. In section 4.2 we briefly discuss the structure of some other amplitudes in this formalism. In section 4.3 we investigate a possible direct regularisation of the MHV Lagrangian.

Treating the polarisation vectors and momenta of the external gluons as four dimensional and lifting the space-time Lagrangian of a massless scalar to D -dimensions, the only place where the -2ϵ dimensional components of the momentum have to be kept is the kinetic term where we replace $\square_D^2 \rightarrow \square + \mu^2$. In one-loop diagrams with a scalar loop gluons only appear in trees glued to the loop. Their momenta will be treated as four-dimensional since they are given by sums of external momenta. Therefore the interaction vertex $\phi^\dagger \overleftrightarrow{\partial}_\mu \phi$ is always contracted with a four-dimensional polarisation vector or momentum so only the four dimensional scalar momentum has to be kept. This leads to a Lagrangian of the form (3.1) with $m^2 = \mu^2$, where loop-integrals are computed using the measure (4.3). The only difference is that the scalars in (3.1) are in the fundamental while the above argument applies to scalars in the adjoint. However, the colour-leading part of amplitudes with adjacent scalars is very easy to lift from fundamental to adjoint representation, the only difference being a factor of N_c . It would also be straightforward to extend the derivation to scalars in the adjoint. Within the twistor approach, the vertex which is needed for scalars in the adjoint is

$$\mathcal{L}_{\text{mass}} = -m^2 \text{tr} \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} \left(H \phi_0^\dagger(x, \pi_1) H^{-1} \right)_1 \left(H \phi_0(x, \pi_2) H^{-1} \right)_2 \quad (4.4)$$

This is obtained by considering the twistor lifting formulae for adjoint scalars. Using the techniques of this article it is easy to see how to obtain the expansion coefficients for adjoint scalars from these if desired. Expanding the frames in the above expression using (2.55) the mass-vertex for scalars in the adjoint becomes

$$V_{\text{CSW}}(B_1, \dots, \xi_i, B_{i+1}, \dots, \xi_j, B_{j+1}, \dots, B_n) = i2^{n/2-1} \frac{m^2 \langle ij \rangle^2}{\langle 12 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle} \quad (4.5)$$

which reduces to (3.5) for scalars at position 1 and n .

There are two remaining issues to show that the CSW rules for massive scalars also work at the quantum level. The first are potential equivalence theorem violations. Their absence for massive scalar amplitudes at tree level and the fact that the four-point all-plus amplitude is reproduced correctly without such contributions (as shown below) suggests that they are also absent at the one-loop level.

The second issue is the absence of anomalies in the Jacobian of the canonical transformation used to derive the CSW-rules. Such anomalies were suspected to be the origin

of the missing rational pieces in pure Yang-Mills amplitudes, but the recovery of the all-plus amplitude from equivalence theorem evasion in [32] suggests that there is no such anomaly in the canonical transformation in the gluon case. The close similarity of the transformation used for the scalars to that of gluons together with our computation of the four-point all-plus amplitude below suggests that there are no such problems in our case as well. Anomalies in the Jacobian would show up as Faddeev-Popov determinants in the twistor action approach. Since an axial gauge choice is employed to derive the rules, ghost contributions are easily seen to be absent.

The reasoning given above uses the fact that the massive scalar Lagrangian (3.1) is appropriate to compute the scalar piece in the SUSY decomposition (4.1) in dimensional regularisation and some evidence that the transformation used to obtain the CSW rules for massive scalars also holds at the quantum level. We would like to point out that a different line of argument can be based on the fact alone that the CSW-rules for a massive scalar give the right tree level amplitudes. The Feynman tree theorem [57] reduces the computation of one-loop amplitudes to dispersion integrals over tree level amplitudes. This has been used to show the η -independence of the one-loop MHV-rules for cut-constructable piece of amplitudes (or amplitudes in supersymmetric theories) from the η -independence of tree-amplitudes [20]. We expect that a similar argument can be given for the massive scalar-loops computed using our CSW rules. Having established η -independence and the resulting absence of spurious poles, one can argue that any unitarity cut of the sum of diagrams will automatically give (dispersion integrals over) on-shell scalar amplitudes. Altogether this discussion strongly suggests that the rules of section 3.1 can be used to compute the scalar-piece in the SUSY decomposition (4.1).

4.1 The four-point all-plus amplitude in pure Yang-Mills

An obvious target for the application of our rules to one-loop diagrams is the all-plus one-loop amplitude which vanishes in supersymmetric gauge theories. As a first example, in this article we will discuss the four-point all-plus amplitude [56]

$$A_4^{\text{scalar}}(g_1^+, g_2^+, g_3^+, g_4^+) = \frac{8i}{(4\pi)^{2-\epsilon}} \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} K_4 \tag{4.6}$$

with

$$K_4 = (-i)(4\pi)^{2-\epsilon} \int \frac{d^4\ell}{(2\pi)^4} \frac{d^{-2\epsilon}\mu}{(2\pi)^{-2\epsilon}} \frac{\mu^4}{d(\ell)d(\ell_1)d(\ell_2)d(\ell_3)} = -\frac{1}{6} + \mathcal{O}(\epsilon) \tag{4.7}$$

We denote the loop momentum flowing between leg 4 and leg 1 by ℓ and use the notation

$$\ell_i = \ell - k_1 - \dots - k_i \quad ; \quad d(\ell_i) = \ell_i^2 - \mu^2 \tag{4.8}$$

The result (4.6) contains a factor 2 taking the two degrees of freedom of a complex scalar into account and differs from the form in [56] by a factor of 4 because of the different colour ordering conventions.

We now show how the result (4.6) for the one-loop four-point amplitude with only positive helicity gluons can be obtained using the diagrammatic rules from section 3.1. The contributing diagrams have the form of box, triangle (T), bubble (B) and tadpole

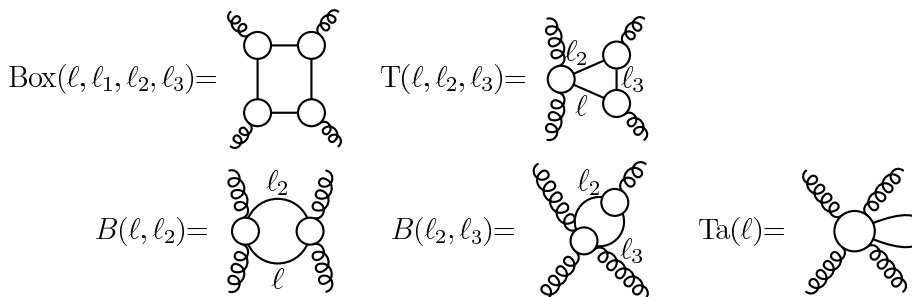


Figure 4: Types of diagrams contributing to the four-point all-plus amplitude

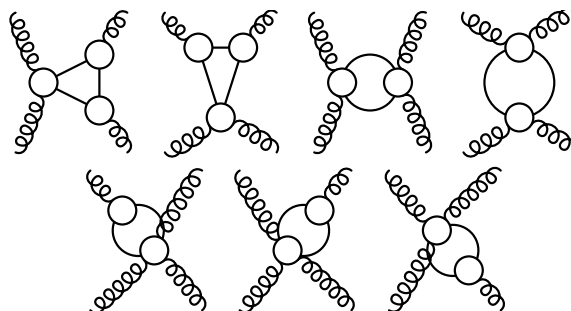


Figure 5: Non-vanishing diagrams contributing to the four-point all-plus amplitude for $|\eta+\rangle = |1+\rangle$

(Ta) topologies, as shown in figure 4. In the diagrams all vertices are given by (3.5). The diagrams are denoted by the momenta flowing through the propagators. The tadpole diagram vanishes because the vertex involves the spinor product $\langle \ell \ell \rangle = 0$ in the numerator. The calculation simplifies if the spinor η used to define the off-shell continuation is fixed in terms of one of the momentum of one of the gluons since this eliminates all diagrams where this gluon enters a three point vertex, cf. the discussion below (3.8). In particular this eliminates the box diagram. For the choice $|\eta-\rangle = |1-\rangle$ one remains with two triangle diagrams ($T(\ell, \ell_2, \ell_3)$, $T(\ell_1, \ell_2, \ell_3)$), two bubble diagrams with four-point vertices ($B(\ell, \ell_2)$, $B(\ell_1, \ell_3)$) and three bubble diagrams with a five-point vertex ($B(\ell_2, \ell_3)$, $B(\ell_1, \ell_2)$ and $B(\ell, \ell_3)$) as shown in figure 5.

Focus first on the two triangle diagrams. Applying the CSW prescription with $|\eta-\rangle = |1-\rangle$ to the loop-momenta we find for the triangle $T(\ell, \ell_2, \ell_3)$

$$\begin{aligned}
 T(\ell, \ell_2, \ell_3) &= \int \frac{d^D \ell}{(2\pi)^D} \frac{(i)^3}{d(\ell)d(\ell_2)d(\ell_2)} \frac{-2i\mu^2 \langle \ell \ell_2 \rangle}{\langle \ell 1 \rangle \langle 1 2 \rangle \langle 2 \ell_2 \rangle} \frac{-\sqrt{2}i\mu^2 \langle \ell_2 \ell_3 \rangle}{\langle \ell_2 3 \rangle \langle 3 \ell_3 \rangle} \frac{-\sqrt{2}i\mu^2 \langle \ell_3 \ell \rangle}{\langle \ell_3 4 \rangle \langle 4 \ell \rangle} \\
 &= \frac{4[12]}{\langle 1 2 \rangle \langle 3 4 \rangle} \int \frac{d^D \ell}{(2\pi)^D} \frac{\mu^4}{d(\ell)d(\ell_2)d(\ell_3)2(\ell \cdot k_1)} \frac{-\mu^2 \langle 1 - |k_3 k_4| 1 - \rangle}{\langle 3 - | \ell_3 | 1 - \rangle \langle 4 - | \ell | 1 - \rangle}
 \end{aligned}
 \tag{4.9}$$

To eliminate the additional factor of μ^2 and the spinor products containing the loop-momentum in the denominator one can use the formula [29]

$$\begin{aligned}
 \mu^2 \langle \eta + |k_i k_{i+1}| \eta - \rangle &= [i(i+1)] \langle \eta + | \ell_i | i + \rangle \langle (i+1) - | \ell_{i+1} | \eta - \rangle \\
 &\quad + d(\ell_{i-1}) \langle \eta + | \ell_i k_{i+1} | \eta - \rangle - d(\ell_i) \langle \eta + | \ell_{i-1} k_{i+1} | \eta - \rangle \\
 &\quad + d(\ell_{i+1}) \langle \eta + | \ell_{i-1} k_i | \eta - \rangle
 \end{aligned}
 \tag{4.10}$$

valid for an arbitrary spinor η and momentum k_i . The labels of the momenta are considered as cyclic, e.g. $\ell_0 \equiv \ell$.

Applying (4.10) to the triangle diagram (4.9) results in one term without spinor products with loop momenta in the denominator and three terms with cancelled propagators:

$$\begin{aligned}
 T(\ell, \ell_2, \ell_3) = \frac{-4[12]}{\langle 12 \rangle \langle 34 \rangle} \int \frac{d^D \ell}{(2\pi)^D} \frac{\mu^4}{d(\ell)d(\ell_2)d(\ell_3)2(\ell \cdot k_1)} \left[[34] + d(\ell_2) \frac{[41]}{\langle 3 - |\ell_3|1- \rangle} \right. \\
 \left. - d(\ell_3) \frac{\langle 1 - |\ell_2|k_{3,4}|1- \rangle}{\langle 3 - |\ell_3|1- \rangle \langle 4 - |\ell|1- \rangle} + d(\ell) \frac{[31]}{\langle 4 - |\ell|1- \rangle} \right] \quad (4.11)
 \end{aligned}$$

Since $2(\ell \cdot k_1) = \ell^2 - \ell_1^2$ the denominator is not yet in the right shape so the first term does not give the correct scalar integral at this stage. The second triangle diagram can be treated in a similar fashion and is given by

$$\begin{aligned}
 T(\ell_1, \ell_2, \ell_3) = \frac{4[41]}{\langle 23 \rangle \langle 41 \rangle} \int \frac{d^D \ell}{(2\pi)^D} \frac{\mu^4}{d(\ell_1)d(\ell_2)d(\ell_3)2(\ell \cdot k_1)} \left[[23] + d(\ell_1) \frac{[31]}{\langle 2 - |\ell_2|1- \rangle} \right. \\
 \left. - d(\ell_2) \frac{\langle 1 - |\ell_1|k_{2,3}|1- \rangle}{\langle 2 - |\ell_2|1- \rangle \langle 3 - |\ell_3|1- \rangle} + d(\ell_3) \frac{[21]}{\langle 3 - |\ell_3|1- \rangle} \right] \quad (4.12)
 \end{aligned}$$

Using momentum conservation to write $[23][41]/\langle 23 \rangle \langle 41 \rangle = [12][34]/\langle 12 \rangle \langle 34 \rangle$ and applying a partial fraction identity

$$\frac{1}{\ell^2 - \ell_1^2} \left[\frac{1}{\ell_1^2 - \mu^2} - \frac{1}{\ell^2 - \mu^2} \right] = \frac{1}{(\ell^2 - \mu^2)(\ell_1^2 - \mu^2)} \quad (4.13)$$

one finds that the sum of the two triangle diagrams produces the correct result (4.6) (up to a factor two since we consider only one scalar degree of freedom) and a sum of terms with cancelled propagators.

The left-over terms are all of the right form to be cancelled by bubble-diagrams and this is indeed what happens. For example the bubble with propagators $d(\ell)$ and $d(\ell_3)$ exactly cancels the corresponding term in (4.11)

$$\begin{aligned}
 B(\ell, \ell_3) &= \int \frac{d^D \ell}{(2\pi)^D} \frac{-4i\mu^2 \langle \ell \ell_3 \rangle}{\langle \ell 1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3 \ell_3 \rangle} \frac{(i)^2}{d(\ell)d(\ell_3)} \frac{-i\mu^2 \langle \ell_3 \ell \rangle}{\langle \ell_3 4 \rangle \langle 4 \ell \rangle} \\
 &= \frac{-4[12]}{\langle 12 \rangle \langle 34 \rangle} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{2(\ell \cdot k_1)d(\ell)d(\ell_3)} \frac{[14]}{\langle 3 - |\ell_3|3- \rangle} \quad (4.14)
 \end{aligned}$$

Similarly the diagram $B(\ell_1, \ell_2)$ cancels the term with missing ℓ_3 propagator in (4.12). The term with missing ℓ_3 propagator in (4.11) is cancelled by a bubble with two four point vertices:

$$\begin{aligned}
 B(\ell, \ell_2) &= \int \frac{d^D \ell}{(2\pi)^D} \frac{-2i\mu^2 \langle \ell \ell_2 \rangle}{\langle \ell 1 \rangle \langle 12 \rangle \langle 2 \ell_2 \rangle} \frac{(i)^2}{d(\ell)d(\ell_2)} \frac{-2i\mu^2 \langle \ell_2 \ell \rangle}{\langle \ell_2 3 \rangle \langle 34 \rangle \langle 4 \ell \rangle} \\
 &= \frac{4[12]}{\langle 12 \rangle \langle 34 \rangle} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{2(\ell \cdot k_1)d(\ell)d(\ell_2)} \frac{\langle 1 + |\ell_2|\ell|1- \rangle}{\langle 3 - |\ell_3|1- \rangle \langle 4 - |\ell|1- \rangle} \quad (4.15)
 \end{aligned}$$

In the same way, the diagram $B(\ell_1, \ell_3)$ cancels the term with missing ℓ_2 propagator in (4.12). The last remaining bubble diagram is given by

$$\begin{aligned}
 B(\ell_2, \ell_3) &= \int \frac{d^D \ell}{(2\pi)^D} \frac{-4i\mu^2 \langle \ell_3 \ell_2 \rangle}{\langle \ell_3 4 \rangle \langle 4 1 \rangle \langle 1 2 \rangle \langle 2 \ell_2 \rangle} \frac{(i)^2}{d(\ell_2) d(\ell_3)} \frac{-i\mu^2 \langle \ell_2 \ell_3 \rangle}{\langle \ell_2 3 \rangle \langle 3 \ell_3 \rangle} \\
 &= \frac{-4[12]}{\langle 1 2 \rangle \langle 3 4 \rangle} \int \frac{d^D \ell}{(2\pi)^D} \frac{1}{d(\ell_2) d(\ell_3)} \frac{[31]^2}{[32] \langle 1 + |\ell_3| 4 + \rangle \langle 2 - |\ell_2| 1 - \rangle}
 \end{aligned} \tag{4.16}$$

This time it requires slightly more work to see the cancellation since both triangle diagrams (4.11) and (4.12) contain a term with ℓ_2 and ℓ_1 propagators. The sum is found to cancel against the the bubble $B(\ell_2, \ell_3)$, as expected:

$$\begin{aligned}
 &[T(\ell, \ell_2, \ell_3) + T(\ell_1, \ell_2, \ell_3)]_{d(\ell_2), d(\ell_3)} \\
 &= -\frac{4[12][34]}{\langle 1 2 \rangle \langle 3 4 \rangle} \int \frac{d^D \ell}{(2\pi)^D} \frac{\mu^4 [31]}{d(\ell_2) d(\ell_3) 2(\ell \cdot k_1)} \left[\frac{1}{\langle 3 + |\ell_2| 1 - \rangle} + \frac{1}{\langle 3 + |\ell_4| 1 - \rangle} \right] \\
 &= \frac{4[12][34]}{\langle 1 2 \rangle \langle 3 4 \rangle} \int \frac{d^D \ell}{(2\pi)^D} \frac{\mu^4}{d(\ell_2) d(\ell_3)} \frac{[31]^2}{\langle 3 + |\ell_2| 1 - \rangle \langle 3 + |\ell_4| 1 - \rangle}
 \end{aligned} \tag{4.17}$$

This completes the demonstration that the result (4.6) is correctly reproduced by the application of the CSW rules for a massive scalar to one-loop diagrams.

Although the number of contributing diagrams is not particularly small, the final result is obtained in a rather transparent way since cancellations among diagrams take place before any loop integral is performed. This is in contrast to the calculation of the all-minus amplitude using light-cone gauge Feynman rules [29] where the final result was obtained after applying extensive Passarino-Veltman reductions to box, triangle and bubble diagrams separately. The essential identity (4.10) always can be applied to products of three-point mass-vertices so the pattern of cancellations is expected to persist in the calculation of more general amplitudes, although a more systematic approach to the combinatorics might be needed. Note however that collinear limits are easily seen to be correctly reproduced in this calculation as these can be read off directly from the vertices. Hence the only ambiguity in the all-multiplicity calculation through massive CSW vertices is the usual ambiguity for the five point amplitude.

4.2 Remarks on other amplitudes

We briefly comment on some features of the calculation of the four-point all plus amplitude give above and on the structure of other amplitudes in the CSW formalism for massive scalars.

The all-plus amplitudes as an anomaly. The finite all-plus amplitude arises from the cancellation of a numerator factor ϵ with an UV $1/\epsilon$ pole. This feature has been used to suggest that this amplitude arises as an anomaly [58]. In our formalism these cancellations are implied by the fact that the mass-vertices (3.5) are proportional to μ^2 and one-loop integrals with an insertion of a power of μ^2 are related to integrals in higher dimensions times explicit pre-factors of ϵ . The situation is somewhat subtle since some factors of

μ^2 might be eliminated through Dirac algebra, c.f. (4.10) in order to cancel some of the unphysical $\frac{1}{\langle np i \rangle}$ singularities in the integrand so that the power-counting in μ^2 is not unique. Nevertheless the calculation in section 4.1 clearly shows how the finite all-plus amplitude arises from the interplay of explicit μ^2 factors supplied by the vertices and the D -dimensional loop-integral measure. Our formalism could also be useful to clarify the anomalous symmetry responsible for the anomaly. A symmetry dimensional regularisation breaks explicitly in our formulation is scaling invariance, and this suggests that this is the anomalously broken symmetry. Note that it cannot be the twistor gauge symmetry as our massive scalar regulator is explicitly invariant under this symmetry. In the canonical formalism this can be interpreted as the absence of Jacobian factors.

The all-minus amplitudes. Using the massive scalar regulator, the all-minus amplitudes are generated only by three-point MHV vertices as in the usual application of the one-loop MHV rules. Note that the propagator is now properly dimensionally regulated. Since in this particular case this calculation is equivalent diagram by diagram to the light-cone calculation it follows that the correct answer will be obtained.

The one-minus amplitudes. The diagrams for amplitudes with one negative helicity gluon contain precisely one ordinary MHV vertex and at least one vertex proportional to μ^2 . This suggests that these amplitudes arise, as in the all-plus case, manifestly as a $\epsilon \times 1/\epsilon$ 'anomaly' in our formalism. For this conclusion it is important that an ordinary MHV vertex with two adjacent legs tied together vanishes. Since this vertex involves a $\frac{1}{\langle 1n \rangle}$ term, at face value this diagram does not vanish, but is actually infinite. However, one should regulate the poles of this particular type by a well-defined prescription [59]. Using the regulator it follows that this particular diagram is a finite tad-pole and hence vanishes in dimensional regularisation.

General structure of the integrands. Note that combining two of our massive scalar vertices into a CSW loop gives almost the same integrand as in the MHV $\mathcal{N} = 4$ case [16]. The difference between the calculations is completely in line with [53]. However, since for massive scalars one is bound to keep the propagator in D dimensions, one cannot simply apply the same reasoning as in [16] which applies dimensional regularisation only after performing the spinor algebra (which is perfectly fine for $\mathcal{N} = 4$). It would be interesting to explore these similarities further since this could lead to a more streamlined evaluation of the integrals encountered here. Combining the rewriting of the loop integrals as dispersion integrals and phase-space integrals of [16] with the formulation of the massive phase-space integral used in [60] could be helpful in this respect.

4.3 Direct construction of pure Yang-Mills amplitudes

We would like to observe that the methods used in the derivation of the massive CSW rules suggest also a more direct way for the quantum completion of the CSW rules independent of the SUSY decomposition. This could be e.g. useful for the application of MHV-methods to two-loop diagrams where a SUSY decomposition is less straightforward. We use the four-dimensional helicity scheme variant of dimensional regularisation since we want to

preserve as much of the four dimensional structure, including supersymmetry, as possible. Therefore we keep the vector fields four-dimensional so that the light-cone vertices can be expressed in terms of the usual spinor products. This is in contrast to [32] for instance, who treat everything in D dimensions. On the level of the action, this is implemented for lightcone Yang-Mills by adding a term,

$$\mathcal{L}_\mu = -2\mu^2 \text{tr} A_z A_{\bar{z}} \tag{4.18}$$

to the Lagrangian. Hence the theory which we'd like to canonically transform contains an extra term which also needs to be transformed. This is of course analogous to the massive scalar case. However, in the present case we also continue the momenta of external gluons to D -dimensions which is in contrast to the computation of the rational piece using a massive scalar loop and massless gluons. Inserting the expansions (2.34) into the above expression will yield a series of terms which are all proportional to μ^2 . Exchanging some of the summations the resulting expression can be brought into the form

$$2\mu^2 \text{tr}[A_{\bar{z},p} A_{z,-p}] = \sum_{n=2}^{\infty} \prod_{i=1}^n \int \widetilde{dk}_i 2(-i)g^n V_{\bar{B}_1, B_2, \dots, B_n} \text{tr}[\bar{B}_1, B_2, \dots, B_n] \tag{4.19}$$

where the vertex is given in terms of the expansion coefficients as

$$\begin{aligned} (-i)g^n V_{\bar{B}_1, B_2, \dots, B_n} &= \mu^2 \sum_{j=1}^{n-1} \sum_{s=1}^j \frac{k_{1+}}{p_+} \mathcal{X}^s(p, k_{n+2-s}, \dots, k_n, k_1, \dots, k_{j+1-s}) \\ &\quad \times \mathcal{Y}(-p, k_{j+2-s}, \dots, k_{n+1-s}) \end{aligned} \tag{4.20}$$

Here the indices have to be interpreted in a cyclic way, i.e $k_{n+1} = k_1$. The quadratic contribution gets re-summed into the $B\bar{B}$ propagator, similar to the massive scalar case.

Inserting the explicit expressions for the expansion coefficients one finds for the three-point vertex

$$\begin{aligned} V_{\bar{B}_1, B_2, B_3} &= (ig^{-3}\mu^2) \left[\mathcal{Y}(k_1, k_2, k_3) + \frac{k_{1,+}}{k_{3,+}} \mathcal{X}^1(k_3, k_1, k_2) + \frac{k_{1,+}}{k_{2,+}} \mathcal{X}^2(k_2, k_3, k_1) \right] \\ &= \frac{\sqrt{2}i\mu^2 \langle \eta 1 \rangle^3}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \left(\frac{\langle 12 \rangle \langle 31 \rangle}{\langle \eta 1 \rangle \langle \eta 2 \rangle \langle 3\eta \rangle} + \frac{\langle 23 \rangle \langle 31 \rangle}{\langle \eta 3 \rangle^2 \langle \eta 2 \rangle} + \frac{\langle 12 \rangle \langle 23 \rangle}{\langle \eta 2 \rangle^2 \langle \eta 3 \rangle} \right) \end{aligned} \tag{4.21}$$

We will now argue that this vertex gives the correct three-point tree-level vertex in the 'four-dimensional' limit $\mu \rightarrow 0$ and consequently the correct quadruple-cut contribution to the four-point all-plus amplitude. Naively, the tree amplitudes generated by the above vertices vanish in the $\mu \rightarrow 0$ limit but one has to be careful about the order of the four-dimensional and the on-shell limits. It is easy to see that in the four dimensional limit there are poles in this expression since in that limit one obtains

$$\langle 12 \rangle [12] \rightarrow 0 \tag{4.22}$$

Hence one should be more careful about the limit. It is not hard to show using the on-shell condition $p_{D,i}^2 = 0 \Rightarrow p_i^2 = \mu^2$ and (2.7) that one obtains exactly the same type of

expression as derived in [32] (in their notation),

$$A = \frac{1^+}{(12)} \left(\frac{p_1^2}{1^+} + \frac{p_2^2}{2^+} + \frac{p_3^2}{3^+} \right) \tag{4.23}$$

Just as in the ‘massive’ scalar case, we arrive at a complete picture of where the missing tree three particle amplitudes are. If one takes the four dimensional limit first and the LSZ reduction next, then these amplitudes arise from equivalence theorem violations. If one takes LSZ first and the four dimensional limit next as in the above example, these amplitudes arise from the dimensional regularisation vertices (4.19). Note that the same restriction to three particle tree amplitudes as in the ‘violations’ case is expected to apply to the tree level contributions generated by (4.19). At the loop level however there is no choice as one needs to integrate over the off-dimensional momenta.

For the all-plus four point amplitude it is easy to see that a quadruple cut of the terms generated by the terms in equation (4.19) will give the same answer as in Yang-Mills theory, since the contributing cut-box diagram only features the on-shell Yang-Mills three point function. Hence it is fully expected that the total expression integrates to the Yang-Mills answer. To proof this one would need to show in addition that triple and double cuts do not give rise to extra contributions.

Let us now briefly comment on amplitudes with more than four legs. For the general case one can use the eikonal identity to write the vertex (4.19) in the form

$$V_{\bar{B}_1, B_2, \dots, B_n} = \frac{-i2^{n/2-1}\mu^2 \langle \eta 1 \rangle^3}{\langle 12 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle} \sum_{i=1}^{n-1} \frac{\langle (i+1)1 \rangle \langle i(i+1) \rangle}{\langle \eta(1+i) \rangle^2 \langle \eta i \rangle} \tag{4.24}$$

There is an analogous twistor expression. This vertex is not yet of the same simplicity as the scalar mass-vertex but recall that here all the gluons are treated as ‘massive’. We conjecture but are as yet unable to proof that these vertices provide a consistent completion of the CSW formalism at the loop level.

As favourable circumstantial evidence this is the case let us note that collinear singularities are manifestly generated by terms in the vertices. Therefore given that the four-point all plus amplitude comes out correctly, the n-particle all plus amplitude will also be contained in the formalism up to the usual ambiguity for the 5 particle amplitude. Furthermore the observations made in section 4.2 in the context of the massive scalar regulator continue to apply, for instance the all-minus amplitude is still given by three-point MHV vertices alone and the all-plus and one-minus amplitudes are expected to arise as a $\epsilon \times 1/\epsilon$ cancellation. Hence our picture of loop level completion is a direct mixture of observations in [32] and [31], with the added bonus of using standard dimensional regularisation and standard vertices. One further difference to [31] is that our regulator is parity asymmetric, while there it is parity symmetric. The major difference to [32] is basically the variant of dimensional regularisation employed: we employ a version of the four dimensional helicity scheme, they treat all their fields in D dimensions.

Finally we would like to remark that the regularisation using the vertex (4.19) should in principle apply also at higher loops. This suggests an interesting picture of higher

loop-amplitudes, where contributions containing the leading $1/\epsilon$ divergences *always* follow from MHV vertices combined into loops, with contributions to sub-leading divergences, finite pieces and higher orders in ϵ arising from the vertices (4.19). This certainly deserves further study, including a more careful treatment of the regularisation then employed here since it is known that dimensional reduction in non-supersymmetric gauge theories requires some additional care at higher loop levels, see e.g. [61].

5. Higgs-gluon couplings

One of the reasons scalar particles are interesting and by extension MHV methods for these is of course the fact that there is a scalar in the standard model: the Higg's particle. Apart from the direct application of our methods to the weak sector, one can study an effective coupling of the Higg's particle to gluons mediated by a top quark loop which will be done in this section.

In the approximation of a large top quark mass, this interaction can be modelled nicely by an effective interaction vertex between Higg's field H and the gluons of the form [62, 63]

$$\frac{1}{2} \frac{\alpha_s}{6\pi v} \int d^4x \tilde{H} F^2 \equiv \frac{1}{2} \int d^4x H F^2. \quad (5.1)$$

with $v \sim 246 \text{ GeV}$ [9]. The above vertex contains of course the simplest possible gauge invariant dimension 5 operator we can construct out of the fields at hand. We have re-absorbed the effective coupling constant into the (now dimensionless) Higg's field H . As advocated in [9] to derive the CSW rules from the space-time action one should split $H = \phi + \phi^\dagger$ and write

$$\int d^4x H F^2 \sim \int d^4x \phi F_+^2 + \phi^\dagger F_-^2 \quad (5.2)$$

The spinor-components of the (anti) self-dual Yang-Mills field strength F_\mp appear in the decomposition

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta} F_{+\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}} F_{-\alpha\beta} \quad (5.3)$$

Using the explicit expression of the symmetrical self-dual field-strength

$$F_{+\dot{\alpha}\dot{\beta}} = \frac{1}{2} (F_{\dot{\alpha}\dot{\beta}}{}^\alpha{}_\alpha) = \frac{1}{2} (\partial_{\dot{\alpha}\alpha} A_{\dot{\beta}}^\alpha + \partial_{\dot{\beta}\alpha} A_{\dot{\alpha}}^\alpha - ig (A_{\dot{\alpha}\alpha} A_{\dot{\beta}}^\alpha + A_{\dot{\beta}\alpha} A_{\dot{\alpha}}^\alpha)) \quad (5.4)$$

it is easy to read off the effective two-gluon ϕ coupling from the expression

$$\frac{1}{4} \phi F_+^2 = \frac{1}{2} \phi \partial_{\dot{\alpha}\alpha} A_{\dot{\beta}}^\alpha (\partial_{\dot{\beta}}^{\dot{\alpha}} A^{\dot{\beta}\beta} + \partial_{\dot{\beta}}^{\dot{\beta}} A^{\dot{\alpha}\beta}) + \dots \quad (5.5)$$

Note that there are only two terms contributing, whereas in the corresponding expression in the four-vector formalism [9] there are three terms including one with an ϵ -tensor. For the amplitude of one ϕ and two on-shell gluons one obtains

$$A_3(\phi, A_1, A_2) = -\frac{i}{2} (p_{1\dot{\alpha}\alpha} \epsilon_{1\dot{\beta}}^\alpha) (p_{2,\dot{\beta}}^{\dot{\alpha}} \epsilon_2^{\dot{\beta}\beta} + p_{2,\dot{\beta}}^{\dot{\beta}} \epsilon_2^{\dot{\alpha}\beta}) = i \langle 1 + |\not{\ell}_1| 2+ \rangle \langle 2 + |\not{\ell}_2| 1+ \rangle \quad (5.6)$$

where the Schouten-identity was used. Inserting the expressions for the polarisation vectors one reproduces in a simple way the known fact that the vertex vanishes as soon as one gluon has positive helicity while for two negative helicity gluon one finds

$$A_3(\phi, A_1^-, A_2^-) = -2i \langle 12 \rangle^2 \quad (5.7)$$

Similarly the MHV-amplitudes with an extra ϕ -particle are just the same as the purely gluonic ones

$$A_{n+1}(\phi, A_1^+, \dots, A_i^-, \dots, A_j^-, \dots, A_n^+) = i2^{n/2-1} \frac{\langle ij \rangle^4}{\langle 12 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle} \quad (5.8)$$

as can be shown using the BCFW relations [64]. In [9] it was proposed to use off-shell continuation of these amplitude as vertices in a CSW-like construction of ϕ +gluon amplitudes. This approach has been extended to amplitudes including quarks [10] and to the one-loop level [64, 21]. In the following we will derive these rules from the action (5.2) using the twistor Yang-Mills approach and will comment on the relation to the canonical approach at the end of this section. In the course of the derivation we obtain additional vertices with an arbitrary number of phi-fields

$$V_n(\phi_1, \dots, \phi_{l-1}, B_l, \dots, \bar{B}_i, \dots, \bar{B}_j, \dots, B_n) = i2^{n/2-1} \frac{\langle ij \rangle^4}{\langle l(l+1) \rangle \dots \langle (n-1)n \rangle \langle nl \rangle} \quad (5.9)$$

While puzzling at first, one can convince oneself that these vertices have to be present in order to calculate amplitudes with more than one Higgs boson. Calculating the four point amplitude with two ϕ fields in ordinary Feynman gauge using the interaction (5.5) one finds that the gluon propagator is cancelled and the same expression as for the three point amplitude is obtained:

$$\begin{aligned} A_4(\phi_1, \phi_2, A_3, A_4) &= -\frac{i}{4} (p_{3,\dot{\alpha}\alpha} \epsilon_{3\dot{\beta}}^\alpha + p_{3\dot{\beta}\alpha} \epsilon_{3\dot{\alpha}}^\alpha) K_\beta^{\dot{\alpha}} \frac{\epsilon^{\beta\gamma} \epsilon^{\dot{\beta}\dot{\gamma}}}{K^2} K_\gamma^{\dot{\delta}} (p_{4,\dot{\delta}\delta} \epsilon_{4\dot{\gamma}}^\delta + p_{4\dot{\gamma}\delta} \epsilon_{4\dot{\delta}}^\delta) \\ &= i \langle 3 + | \not{\epsilon}_3 | 4 + \rangle \langle 4 + | \not{\epsilon}_4 | 3 + \rangle \end{aligned} \quad (5.10)$$

This type of cancellation can be shown to hold even off-shell. It therefore applies directly to the full tower of amplitudes with two gluons and the rest ϕ particles. A BCFW type argument as in [64] then yields the scattering amplitudes of MHV type easily. By extension, it is then also easy to conjecture that these extra MHV amplitudes can be promoted to MHV vertices.

5.1 Twistor Yang-Mills derivation

The twistor lifting formulae of section 2.3 (and a sneak peek at the known result) indicate that we'd rather consider terms like

$$\sim \phi C^2 \quad (5.11)$$

added to the Chalmers and Siegel action (2.40) as these lift easily to twistor space. As we have a field equation equating $C = F^+$ in the Chalmers and Siegel action this is also a quite plausible term. Integrating out C from

$$\int d^4x C F_+ - \frac{1}{2} C^2 + \frac{1}{2} \phi C^2 \quad (5.12)$$

however does not yield the action we are interested in. Rather, this yields

$$\frac{1}{2} \int d^4x \frac{1}{1-\phi} (F_+)^2 \tag{5.13}$$

where the inverse is defined by it's Taylor series. Instead, one has to integrate out C from

$$\int d^4x C F_+ - \frac{1}{2} C^2 + \frac{1}{2} \frac{\phi}{1+\phi} C^2 \tag{5.14}$$

to obtain the action of interest in this section. In these computations a field dependent determinant arises which however does not contribute to scattering amplitudes when calculated in dimensional regularisation since it involves a trivial kinetic term.

The action thus obtained can be lifted easily to twistor space by the same formulae as in 2.3. Since the Higgs is not charged under the strong gauge group, it can either be left on space-time or simply be lifted by

$$\phi(x) = \int dk \xi_0 \tag{5.15}$$

without any of the strong gauge group frames. In particular, the coupling of the Higgs fields ξ to the twistor field \bar{B}_0 is given by

$$- \text{tr} \int d^4x dk_1 dk_2 \frac{\int \xi_0}{1 + \int \xi_0} \langle \pi_1 \pi_2 \rangle^2 (H \bar{B}_0 H^{-1})_{(1)} \wedge (H \bar{B}_0 H^{-1})_{(2)}. \tag{5.16}$$

Expanding out the inverse yields a sum over terms with increasing numbers of ξ particles.

In the obtained twistor action two different gauges are again available as the entire formalism is invariant under the appropriate gauge group by construction. Picking space-time gauge will reduce the action down to the space-time action we started of from. However, the real fun begins when one selects CSW gauge. It is not hard to see that the structure of the vertices in 5.16 are all the same as the ' C^2 ' term in Yang-Mills theory. Hence it is also not hard to see that in CSW gauge the entire tower of vertices generates scattering amplitudes of the same MHV form as in pure gauge theory, the only difference being the momentum conservation delta function which involves all participating particles, including the Higgs particles. Furthermore, the MHV rules are also easy to derive by the same analogy and are exactly those given in [9] in the case of one Higg's particle. As a special case one obtains the vanishing of the equal helicity and one-unequal helicity Higgs-gluon amplitudes.

What is new in the above derivation is that there is actually a tower of ' ϕ^i glue' amplitudes, all of the same MHV form. It is not known yet how practical this is as for instance the H^2 amplitudes would also involve calculating a $\phi\phi^\dagger$ amplitude. In addition, other effects than the effective interaction considered here might start dominating when one considers 2-Higgs processes. One such effect is the contribution of a top quark loop with two Higgs particles attached that leads to an $F_{\mu\nu} F^{\mu\nu} H^2$ term in the effective action. Of course, what is easy to calculate is the case of gluon amplitudes with one pseudo-scalar Higgs ($\sim \phi - \phi^\dagger$) and one Higgs ($\sim \phi + \phi^\dagger$) since this requires only $\phi^2 + g$ vertices and the complex conjugate. In fact, many MHV-type calculations for this case will probably only require minimal effort once the single Higg's calculation has been done.

5.2 Extension to more general vertices

The arguments above generalise easily to one particular class of space-time vertices. Consider instead of the coupling studied above a scalar field coupled through a higher dimension operator,

$$\mathcal{L}_{\text{int}} = \kappa \text{tr} \phi(x) (F_+)^i \quad (5.17)$$

for some natural number i with some particular contraction amongst the indices of F . The field ϕ can be charged or uncharged under the gauge group. This type of operator arises for instance in studying sub-leading effects in the effective action (5.1) suppressed by powers of the top quark mass. As above, in order to lift this easily to twistor space, F_+ needs to be replaced by the Chalmers and Siegel C field. It is not hard to guess that one obtains an infinite tower of couplings with increasing numbers of ϕ fields. Lifting these to twistor space and choosing CSW gauge then easily generates towers of MHV vertices and scattering amplitudes. In particular, the interaction written above will generate amplitudes which involve at least i negative helicity gluons, while for precisely i the amplitudes will be holomorphic. The contractions between indices in the higher dimension operator translates into a particular contraction of the spinor momenta of the negative helicity gluons in the numerator. A Noether argument will then also yield couplings of the ϕ field to quarks or scalars in any representation if so desired.

As an example of the above general argument, consider the term

$$\mathcal{L}_{\text{int}} = \kappa \phi \text{tr} \left(F_{\dot{\alpha}}^{\dot{\beta}} F_{\dot{\beta}}^{\dot{\gamma}} F_{\dot{\gamma}}^{\dot{\alpha}} \right) \quad (5.18)$$

A complication with respect to the quadratic analysis is the fact that the cubic interaction cannot be integrated out exactly. However, as we are only interested here in perturbation theory, this obstacle can be overcome. For this it is easier to consider the inverse problem,

$$\begin{aligned} Z[J] &= \int dC dA e^{\int C_{\dot{\alpha}\dot{\beta}} F^{\dot{\alpha}\dot{\beta}} - \frac{1}{2} C_{\dot{\alpha}\dot{\beta}} C^{\dot{\alpha}\dot{\beta}} + \kappa \phi(x) \text{tr} \left(C_{\dot{\alpha}}^{\dot{\beta}} C_{\dot{\beta}}^{\dot{\gamma}} C_{\dot{\gamma}}^{\dot{\alpha}} \right) + J_{\dot{\alpha}\dot{\beta}} C^{\dot{\alpha}\dot{\beta}}} \\ &= \int dC dA \sum_{i=0}^{\infty} \frac{1}{i!} \left(\int \kappa \phi \frac{\partial}{\partial J} \frac{\partial}{\partial J} \frac{\partial}{\partial J} \right)^i e^{\int C_{\dot{\alpha}\dot{\beta}} F^{\dot{\alpha}\dot{\beta}} - \frac{1}{2} C_{\dot{\alpha}\dot{\beta}} C^{\dot{\alpha}\dot{\beta}} + J_{\dot{\alpha}\dot{\beta}} C^{\dot{\alpha}\dot{\beta}}} \\ &= \int dA \sum_{i=0}^{\infty} \frac{1}{i!} \left(\int \kappa \phi \frac{\partial}{\partial J} \frac{\partial}{\partial J} \frac{\partial}{\partial J} \right)^i e^{\int (F_+ + J)^2} \end{aligned} \quad (5.19)$$

At this point some thought is needed. First of all, if the J^2 term was absent, then the result would be obvious. It is also clear that the case considered in the first part of this section one obtains increasing numbers of ϕ fields as in (5.13): the J^2 term is a 'unity' propagator. Actually, this simply echoes the result in (5.10). This propagator cannot connect a vertex to itself as this would yield a tadpole loop contribution of the form

$$\sim \int d^d p \quad (5.20)$$

which vanishes in dimensional regularisation. Restricting to tree level one finds that ' J^2 ' always must connect two different vertices for a non-trivial result and therefore increases the number of external ϕ and C fields.

With this analysis the Lagrangian in terms of mostly C fields which reproduces (5.18) can be constructed perturbatively,

$$S = \int \int C_{\dot{\alpha}\dot{\beta}} F^{\dot{\alpha}\dot{\beta}} - \frac{1}{2} C_{\dot{\alpha}\dot{\beta}} C^{\dot{\alpha}\dot{\beta}} + \kappa \phi(x) \text{tr} \left(C_{\dot{\alpha}}^{\dot{\beta}} C_{\dot{\beta}}^{\dot{\gamma}} C_{\dot{\gamma}}^{\dot{\alpha}} \right) + \text{terms with more } \phi, C\text{'s} \quad (5.21)$$

This can be lifted to twistor space. It is furthermore easy to see that in CSW gauge the scattering amplitude with 3 negative and an arbitrary number of positive helicity gluons generated by (5.18) is

$$\langle \phi g_1^+ \dots g_r^- \dots g_s^- \dots g_t^- \dots g_n \rangle = i\sqrt{2}^n \kappa \frac{\langle rs \rangle^2 \langle st \rangle^2 \langle tr \rangle^2}{\langle 12 \rangle \dots \langle n1 \rangle} \quad (5.22)$$

confirming an expression in [9].

More general cases than the example considered here follow easily. For combinations of quadratic vertices with any number of higher dimensional operators of the form (5.18) it is convenient to perform the path integral over the quadratic ones exactly. This situation occurs for instance when one wants to study sub-leading terms in the effective action for Higg's particles coupled to glue through a massive top quark loop.

5.3 Coupling in other matters

In [10] the extension of the above analysis in the quadratic case to include Higg's particle-quark interactions was discussed. In the context of the twistor action approach to gauge theory it is now very simple to derive these results directly from the action as well. The main observation needed is that within the lifting approach to the gauge theory one should keep at all times the extra twistor space gauge symmetry manifest. In the glue coupled to matter case, this requirement was shown in [28] to lead to towers of vertices via the Noether procedure. These towers translate to CSW-style scattering amplitudes and vertices. For $\mathcal{N} = 1$ for instance, this will reproduce exactly the gluon-gluino and gluino-gluino couplings needed for the appropriate CSW rules in this case. For $\mathcal{N} = 4$, application of the same argument reproduces Nair's super-vertex [3] in the action. It is then a simple matter to extend the same type Noether calculation to the vertex leading to (5.16). It is again important here that the Higgs is uncharged under the strong gauge group. Hence in the case of the vertex (5.1) the CSW formalism as described in [10] is easily derived with the added bonus of having an arbitrary number of ϕ particles. Furthermore, this derivation can be made to work directly for matter transforming in the fundamental. A worked out example of applying the Noether procedure to the twistor action can be found in appendix D.

One point to bear in mind in all this is that the generated MHV vertices are derived for a particular space-time action which can be re-obtained by imposing space-time gauge. As a consequence, just as noted before, there is an ambiguity for deriving the couplings of colored scalars since there is more than one gauge-invariant term on twistor space for four colored scalars.⁹ On space-time this corresponds to the effective dimension 5 vertex,

$$\mathcal{L}_{\text{ambiguous}} \sim \int \phi(x) \text{tr} (s\bar{s})^2 \quad (5.23)$$

⁹One can show there is no such term for the fermions.

where s and \bar{s} denote the space-time scalar fields. In principle the value of this coupling would have to be calculated separately. However, due to the lack of scalar fields charged under the strong gauge group in the standard model this ambiguity is at present of mostly theoretical value.

The analysis for the general class of higher dimensional operators follows along similar lines. In terms of the example studied above, starting from the twistor lift of (5.21) one obtains twistor vertices of the form,

$$V_{\text{CSW}}(\bar{\lambda}_1 B_2, \dots, \bar{B}_i \dots \bar{B}_j \dots B_{n-1} \lambda_n) = 6\kappa\phi(x) \frac{1}{2} \int_{(\mathbb{CP}^1)^4} \frac{\langle \pi_1 \pi_2 \rangle \langle \pi_2 \pi_3 \rangle \langle \pi_3 \pi_1 \rangle}{\langle \pi_1 \pi_4 \rangle} (\bar{\lambda}_0 H^{-1})_{(1)} (H \bar{B} H^{-1})_{(2)} (H \bar{B} H^{-1})_{(3)} (H \lambda_0)_{(4)} \quad (5.24)$$

$$V_{\text{CSW}}(\bar{\lambda}_1 B_2, \dots, \lambda_i \bar{\lambda}_{i+1} B_{i+2} \dots \bar{B}_j \dots B_{n-1} \lambda_n) = 6\sqrt{2}\kappa\phi(x) \frac{1}{2} \int_{(\mathbb{CP}^1)^5} \frac{\langle \pi_1 \pi_2 \rangle \langle \pi_2 \pi_4 \rangle \langle \pi_4 \pi_1 \rangle}{\langle \pi_3 \pi_4 \rangle \langle \pi_5 \pi_1 \rangle} (\bar{\lambda}_0 H^{-1})_{(1)} (H \bar{B} H^{-1})_{(2)} (H \lambda_0)_{(3)} (\bar{\lambda}_0 H^{-1})_{(4)} (H \lambda_0)_{(5)} \quad (5.25)$$

$$V_{\text{CSW}}(\bar{\lambda}_1 B_2, \dots, \lambda_i \bar{\lambda}_{i+1} B_{i+2} \dots \lambda_j \bar{\lambda}_{j+1} B_{j+2}, \dots, \lambda_n) = 6\kappa\phi(x) \frac{1}{2} \int_{(\mathbb{CP}^1)^6} \frac{\langle \pi_1 \pi_3 \rangle \langle \pi_3 \pi_5 \rangle \langle \pi_5 \pi_1 \rangle}{\langle \pi_2 \pi_3 \rangle \langle \pi_4 \pi_5 \rangle \langle \pi_6 \pi_1 \rangle} (\bar{\lambda}_0 H^{-1})_{(1)} (H \lambda_0)_{(2)} (\bar{\lambda}_0 H^{-1})_{(3)} (H \lambda_0)_{(4)} (\bar{\lambda}_0 H^{-1})_{(5)} (H \lambda_0)_{(6)} \quad (5.26)$$

with λ and $\bar{\lambda}$ a positive and negative helicity quark transforming in the fundamental respectively. These vertices have to be integrated over space-time. It is straightforward to derive scattering amplitudes from these vertices in CSW gauge. In addition, more general vertices and more general forms of matter are a straightforward extension.

5.4 Charged scalar vertices

Although of less phenomenological relevance presently, it is easy to see that the methods in this section extend also to non-minimally coupled colored scalars transforming in the adjoint or in the 'bi-fundamental, bi-anti-fundamental' representation. The difference is that the twistor lift of

$$\text{tr } \phi F_{\pm}^2 \quad (5.27)$$

in the case of an adjoint scalar after an analogous analysis now reads

$$\text{tr} \int dk_1 dk_2 dk_3 (H \xi_0 H^{-1})_1 (H \bar{B}_0 H^{-1})_2 (H \bar{B}_0 H^{-1})_3 \langle \pi_2 \pi_3 \rangle^2 \quad (5.28)$$

In addition, there are analogous terms with more ϕ fields inserted. The above term will lead to holomorphic scattering amplitudes of the form

$$A(\phi_1 B_2 \dots \bar{B}_i \dots \bar{B}_j \dots) \sim \frac{\langle \pi_1 \pi_i \rangle \langle \pi_i \pi_j \rangle^3 \langle \pi_j \pi_1 \rangle}{\langle 12 \rangle \dots \langle n1 \rangle} \quad (5.29)$$

and, more generally, to MHV vertices of the indicated shape. Coupling in additional matters is more involved in this particular case since the scalar is charged under the gauge group.

5.5 Comment on canonical transformation method

In principle the above twistor derivation of the Higgs-gluon couplings can be derived using the canonical transformation. This however turns out to be fairly tedious and, in keeping with our general observations on the applicability of CSW-rules- derivation methods, *if* the twistor approach works it will generate complete vertices at once. For comparison reasons however, let us derive here the E-M transformation coefficients for the case of the twistor lift of the action in equation (5.14).

The twistor calculation uses the same gauge transformation as before to trivialise the self-dual Yang-Mills vertex. Therefore the transformation for A_z is also given by the same coefficients. In the twistor derivation of the E-M coefficients however, the field equation for \bar{B}_0 was used. As there is in the case at hand an extra tower of terms which depends on this field, the E-M coefficients will be changed. However, it is also easy to see that the change will only be by a multiplicative function,

$$A_z = A_z^{\text{pure glue}} \left(\frac{1}{1 + \phi} \right). \quad (5.30)$$

We expect, but do not attempt to prove here, that the above canonical transformations would arise in an attempt to derive the CSW rules from the lightcone Yang-Mills theory with an effective interaction (5.1). One immediate problem one has to face there is that integrating out the A_- component is complicated by this interaction, for instance due to derivatives of the ϕ field generated by integrating by parts. Coupling in matter fields seems to be even a step beyond that. On the other hand, if one would be able to apply a canonical transformation to the Chalmers-Siegel action, clearly the same results as from the twistor approach would be obtained.

6. Conclusions

In this article several aspects of deriving CSW rules for massive matter have been discussed. As an example, we focused on scattering of massive scalars, but we expect that our methods can be applied to much wider classes of interesting gauge theories and to a wider class of questions within these. As another application of deriving CSW rules we have shown how all CSW rules which involve an effective Higgs-gluon coupling studied in the literature can be easily reproduced from the action. In addition, we have found some CSW vertices with more than one Higgs boson that have not been studied so far in the literature. It would be interesting to see if these extra vertices can be applied to phenomenologically interesting amplitudes such as multi-gluon amplitudes with one pseudo-scalar Higgs and one Higgs.

From the phenomenological point of view, an immediate next step would be the extension to amplitudes with top quarks. Another interesting avenue to pursue would be to study questions on electroweak amplitudes that are especially relevant for applications at the LHC (see e.g. [55]). In particular, it would be interesting to see if the enlarged gauge freedom of the twistor action allows to choose a gauge which simplifies the electroweak action beyond what is currently used. Since the CSW rules can be used to simplify the proof

of the BCFW on-shell recursions, extensions of our approach to massive particles with spin could also provide further insight into their treatment in the BCFW approach [40–43].

On the methodical side, our analysis has shown how the canonical transformation method and the twistor action method are related by deriving the transformation formulae of one from the other. We found it interesting to see how two completely differently motivated derivations can lead to the same result. Both methods have their own advantages and disadvantages, which we elaborated upon already earlier. It would be desirable to obtain a more systematical understanding of the canonical transformation method in order to allow a simpler derivation of the expansion coefficients without the need to solve recursion relations. The most pressing issue from the twistor point of view is to find a twistor action for Einstein gravity. Since there is a close relationship between twistor methods and harmonic superspace methods (see also [30]) and actions have been constructed for harmonic superspace supergravity, it is not hard to imagine a similar reasoning will lead to a twistor supergravity action. This would be very interesting to obtain, as the amplitudes generated by this action should give some form of the CSW rules for gravity.

Yet another set of questions raised by this article concerns loop effects in pure glue Yang-Mills theory. As shown, the massive scalar CSW rules can be used to calculate one loop amplitudes at least in principle, and it would be interesting to see if the computation can be streamlined more, both for more external particles as for addition of more 'minus' gluons. In addition, the rules are compatible with the proposal that some amplitudes arise as an anomaly in dimensional regularisation and suggest this is an anomaly in the conformal symmetry. Motivated by the massive CSW rules we initiated the study of a direct regulator of the pure glue Yang-Mills Lagrangian that, at least in principle, could extend to higher loop computations. However, there are still open questions that have to be clarified, such as finding a more compact expression for the regulating terms and verifying the consistency of the proposed regularisation scheme.

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A. Yang-Mills conventions

For the generators T^a of the fundamental representation of $SU(N)$ we use the normalisation

$$\text{tr } T^a T^b = \frac{1}{2} \delta^{ab} \quad (\text{A.1})$$

and the Lie algebra $[T^a, T^b] = if^{abc}T^c$. One can translate to these conventions by replacing $T^a \rightarrow (-i)T^a$ in [44] while in comparison to [29, 31] one needs to replace $T \rightarrow \sqrt{2}T$. The Yang-Mills Lagrangian is

$$\mathcal{L} = -\frac{1}{2} \text{tr}[F^{\mu\nu} F_{\mu\nu}] \quad (\text{A.2})$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ and $A_\mu = T^a A_\mu^a$. Expressions from [44] can be converted to these conventions by replacing $A_\mu \rightarrow -igA_\mu$.

For amplitudes with scalars transforming under the fundamental representation of $SU(N)$ we employ the usual decomposition of the full amplitude \mathcal{A}_n into gauge invariant partial amplitudes A_n defined by [2, 39]:

$$\mathcal{A}_n(\phi_1, g_2, g_3, \dots, g_{n-1}, \phi_n) = g^{n-2} \sum_{\sigma \in S_{n-2}(2, \dots, n-1)} (T^{a_{\sigma(2)}} \dots T^{a_{\sigma(n-1)}})_{ij} A_n(\phi_1, g_{\sigma(2)}, \dots, g_{\sigma(n-1)}, \phi_n). \quad (\text{A.3})$$

where the sum is over all permutations of the external gluon legs.

B. Spinor identities

A useful consequence of the Schouten identity $0 = \langle ab \rangle \langle cd \rangle + \langle ac \rangle \langle db \rangle + \langle ad \rangle \langle bc \rangle$ is the so called eikonal identity (see e.g. [39])

$$\sum_{i=j}^{k-1} \frac{\langle i(i+1) \rangle}{\langle i\eta \rangle \langle \eta(i+1) \rangle} = \frac{\langle jk \rangle}{\langle j\eta \rangle \langle \eta k \rangle} \quad (\text{B.1})$$

As an application of the eikonal identity one can derive the formula

$$\sum_{i=j}^{n-1} \frac{(k_{j,i})_+ \langle i(i+1) \rangle}{\langle i\eta \rangle \langle \eta(i+1) \rangle} = \frac{\langle \eta + |k_{j,n-1}|n+ \rangle}{2 \langle \eta n \rangle} \quad (\text{B.2})$$

where we have used the translation between light-cone components and spinor-products to write $2(k_k)_+ \langle kn \rangle = \langle k\eta \rangle \langle \eta + |k_k|n+ \rangle$. As a generalisation one has (for $l < j$)

$$\sum_{i=j}^{n-1} \frac{(k_{l,i})_+ \langle i(i+1) \rangle}{\langle i\nu \rangle \langle \nu(i+1) \rangle} = \frac{\langle \nu + |k_{j,n-1}|n+ \rangle}{2 \langle \nu n \rangle} + \frac{(k_{l,j-1})_+ \langle jn \rangle}{\langle j\nu \rangle \langle \nu n \rangle} \quad (\text{B.3})$$

C. Derivation of the transformation of $A_{\bar{z}}$ in the canonical method

In this appendix we show that the transformations of the scalars (3.29) and (3.33) together with the additional piece in the transformation of the gluon momentum (3.35) leave the sum of scalar and gluon kinetic terms invariant, c.f. (3.34).

The kinetic term of the scalars alone is not transformed just into a quadratic term but into a sum of terms

$$(p_+ \bar{\phi}_p)(p_- \phi_{-p}) = \sum_{n=2}^{\infty} \int \prod_{i=1}^n \widetilde{dk}_i (g\sqrt{2})^{n-2} \frac{\mathcal{K}_{k_1, \dots, k_n}^\phi \langle \eta 1 \rangle \langle \eta n \rangle}{\langle 12 \rangle \dots \langle (n-1)n \rangle} (\bar{\xi}_{k_1} B_{k_2} \dots B_{k_{n-1}} \xi_{k_n}) \quad (\text{C.1})$$

where the coefficients simplify using the eikonal identity and (B.2)

$$\begin{aligned} \mathcal{K}_{k_1, \dots, k_n}^\phi &= \sum_{j=1}^{n-1} \frac{(k_{1,j})_+ (k_{j+1,n})_- \langle j(j+1) \rangle}{\langle \eta j \rangle \langle \eta(j+1) \rangle} = \sum_{l=2}^n \sum_{k=1}^{l-1} \frac{(k_k)_+ (k_l)_- \langle kl \rangle}{\langle \eta l \rangle \langle \eta k \rangle} \\ &= \sum_{l=2}^{n-1} (k_l)_- \frac{2 \langle \eta + |k_{1,l-1}| l + \rangle}{2 \langle \eta l \rangle} \end{aligned} \quad (\text{C.2})$$

The term $l = n$ has been dropped since it vanishes due to momentum conservation. While the $n = 2$ term in (C.1) gives the correct kinetic Lagrangian for the ξ scalars, the terms with $n \geq 3$ have to be cancelled by the scalar contribution to $\partial_+ A_{\bar{z}}$ in the transformation of the kinetic term of the gluons. For this we make an Ansatz of the form

$$p_+(A_{p,\bar{z}})_{ij} |_{\bar{\xi}\xi} = \frac{1}{2} \sum_{n=2}^{\infty} \sum_{s=1}^{n-1} \int \prod_{i=1}^n \widetilde{dk}_i \mathcal{W}^s(p, k_1, \dots, k_n) (k_{1+\bar{\xi}-k_1} \dots B_{-k_s})_i (B_{-k_{s+1}} \dots \xi_{-k_n})_j \quad (\text{C.3})$$

Here the group-index structure has been made explicit. Inserting the expansions of A_z and $A_{\bar{z}}$ from (2.30) and (C.3) into the kinetic term of the gluons leads to the expression

$$\begin{aligned} 2 \text{tr}[(p_+ A_{p,\bar{z}})(p_- A_{-p,z})] &= 2 \text{tr}[(p_+ \bar{B}_p)(p_- B_{-p})] + \sum_{n=3}^{\infty} \sum_{s=1}^{n-2} \sum_{j=s+1}^{n-1} (\bar{\xi}_{k_1} B_{k_2} \dots B_{k_{n-1}} \xi_{k_n}) \\ &\quad (k_{1+})_+ (k_{s+1,j})_- \mathcal{W}^s(k_{s+1,j}, k_1, \dots, k_s, k_{j+1}, \dots, k_n) \mathcal{Y}(-k_{s+1,j}, k_{s+1}, \dots, k_j) \end{aligned} \quad (\text{C.4})$$

The coefficients of a given $(k_l)_-$ must cancel between (C.4) and (C.1). For instance, the cancellation of the coefficient of $(k_2)_-$ against the $n = 3$ term in (C.1) determines the first term in the Ansatz (C.3):

$$\mathcal{W}^1(k_2, k_1, k_3) = -(g\sqrt{2}) \frac{\langle \eta 1 \rangle \langle \eta 3 \rangle \langle \eta + |k_1| 2 + \rangle}{\langle \eta 2 \rangle \langle 12 \rangle \langle 23 \rangle 2(p_1)_+} = (g\sqrt{2}) \frac{\langle \eta 3 \rangle}{\langle \eta 2 \rangle \langle 23 \rangle} = (g\sqrt{2}) \frac{\langle \eta 3 \rangle}{\langle \eta 1 \rangle \langle 31 \rangle} \quad (\text{C.5})$$

where momentum conservation and the identity (2.7) were used in the last step. With some more work one determines the next coefficients in the Ansatz from the cancellation of the terms proportional to $(k_2)_-$ and $(k_3)_-$ in the $n = 4$ terms as

$$\mathcal{W}^1(p, k_1, k_2, k_3) = 2g^2 \frac{\langle \eta 3 \rangle^2}{\langle \eta 1 \rangle \langle \eta 2 \rangle \langle 23 \rangle \langle 31 \rangle}; \quad \mathcal{W}^2(p, k_1, k_2, k_3) = 2g^2 \frac{\langle \eta 3 \rangle}{\langle \eta 2 \rangle \langle 12 \rangle \langle 31 \rangle} \quad (\text{C.6})$$

These explicit results motivate an all-multiplicity Ansatz for the coefficients \mathcal{W} :

$$\mathcal{W}^s(p, k_1, \dots, k_n) = (g\sqrt{2})^{n-1} \frac{\langle \eta n \rangle^2 \langle s(s+1) \rangle}{\langle \eta s \rangle \langle \eta(s+1) \rangle \langle 12 \rangle \dots \langle (n-1)n \rangle \langle n1 \rangle} \quad (\text{C.7})$$

After some relabelling one obtains the form quoted in (3.35). Inserting the result for \mathcal{Y} and the Ansatz for \mathcal{W} into (C.4) results in an expression of the same form as (C.1) with \mathcal{K}^ϕ replaced by

$$\mathcal{K}_{k_1, \dots, k_n}^A = \frac{\langle \eta 1 \rangle \langle \eta n \rangle}{\langle n1 \rangle} \sum_{l=2}^{n-1} (k_l)_- \sum_{s=1}^{l-1} \sum_{j=l}^{n-1} \frac{(k_{s+1,j})_+ \langle s(s+1) \rangle \langle j(j+1) \rangle}{\langle \eta s \rangle \langle (s+1)\eta \rangle \langle \eta j \rangle \langle (j+1)\eta \rangle} \quad (\text{C.8})$$

Here the order of the summations was exchanged in order to isolate the coefficient of a given $(k_l)_-$. Performing the summation over j using (B.3) leads to the identity required for the cancellation of the $n \geq 3$ terms in (C.2):

$$\begin{aligned} \mathcal{K}_{k_1, \dots, k_n}^A &= \frac{\langle \eta 1 \rangle}{\langle n 1 \rangle} \sum_{l=2}^{n-1} (k_l)_- \sum_{s=1}^{l-1} \frac{\langle s(s+1) \rangle}{\langle \eta s \rangle \langle (s+1)\eta \rangle} \left(\frac{1}{2} \langle \eta + |k_{l,n}|n+ \rangle + (k_{s+1, l-1})_+ \frac{\langle l n \rangle}{\langle l \eta \rangle} \right) \\ &= \sum_{l=2}^{n-1} \frac{(k_l)_-}{2 \langle l \eta \rangle \langle n 1 \rangle} (\langle \eta + |k_{1, l-1}|n+ \rangle \langle l 1 \rangle + \langle \eta + |k_{2, l-1}|1+ \rangle \langle n l \rangle) = -\mathcal{K}_{k_1, \dots, k_n}^\phi \end{aligned} \quad (\text{C.9})$$

Here $k_{i,j}$ is understood to be zero for $j < i$. Momentum conservation and the eikonal identity have been used in the second step while the last step uses the Schouten identity. This concludes the demonstration that the sum of the kinetic terms (3.34) stays invariant and the transformations (3.29) and (3.35) are indeed canonical.

D. Harmonic expansions and Noether procedure for actions on twistor space

In this appendix we show that given the twistor action for pure Yang-Mills theory,

$$\begin{aligned} S_{\text{YM}} &= 2 \text{tr} \int d^4 x k \bar{B} \wedge (\bar{\partial} B - i\sqrt{2}g B \wedge B) \\ &\quad - \text{tr} \int d^4 x k_1 k_2 \wedge \langle \pi_1 \pi_2 \rangle^2 (H \bar{B} H^{-1})_{(1)} \wedge (H \bar{B} H^{-1})_{(2)} \end{aligned} \quad (\text{D.1})$$

which was constructed in section 2.3 the action for a massive scalar coupled to Yang-Mills follows by the Noether procedure. In the course of the derivation we will derive the twistor lifting formula for scalars used in the main text. The Noether procedure will of course need a free action as an input. It is easy to verify that one should consider

$$S_{\text{scalar}} = \int \Omega \wedge \bar{\xi} \wedge \bar{\partial} \xi \quad (\text{D.2})$$

for two as yet uncharged weight -2 twistor $(0, 1)$ form fields $\bar{\xi}, \xi$. There are two related reasons for this. The more abstract one is that the Penrose transform will relate cohomology classes on twistor space with solutions to the field equations on space-time. The field equations of the above action give exactly those classes, since it has a symmetry

$$\bar{\xi} \rightarrow \bar{\xi} + \bar{\partial} \bar{f}^{-2} \quad \xi \rightarrow \xi + \bar{\partial} f^{-2} \quad (\text{D.3})$$

with f^{-2} and \bar{f}^{-2} scalar functions of the indicated weight so solutions to the field equations

$$\bar{\partial} \bar{\xi} = \bar{\partial} \xi = 0 \quad (\text{D.4})$$

up to gauge equivalence are the definition of the Dolbeault cohomology class. The more concrete reason (D.2) is the correct action to consider is that the gauge symmetry (D.3) can be fixed to 'space-time gauge',

$$\partial_0^\dagger \xi_0 = \partial_0^\dagger \bar{\xi}_0 = 0 \quad (\text{D.5})$$

Solutions to these conditions are harmonic functions on the sphere, so

$$\xi_0 = \phi(x) \quad \bar{\xi}_0 = \bar{\phi}(x) \tag{D.6}$$

by a straightforward cohomology argument on the Riemann sphere for fields of this specific weight. The fields ξ_α and $\bar{\xi}_\alpha$ can be integrated out to yield

$$\xi_\alpha = \frac{\hat{\pi}^{\dot{\alpha}}}{\langle \pi \hat{\pi} \rangle} \partial_{\alpha\dot{\alpha}} \phi(x) \tag{D.7}$$

$$\bar{\xi}_\alpha = \frac{\hat{\pi}^{\dot{\alpha}}}{\langle \pi \hat{\pi} \rangle} \partial_{\alpha\dot{\alpha}} \bar{\phi}(x) \tag{D.8}$$

Plugging back into the action this gives

$$\int d^4x \partial_{\alpha\dot{\alpha}} \bar{\phi}(x) \partial^{\alpha\dot{\alpha}} \phi(x) \tag{D.9}$$

which is the free field Lagrangian. This is basically the argument in [33].

Harmonic expansions and the twistor transform. Another way to understand the gauge fixing argument is through a harmonic expansion on the Riemann sphere of the weight 0 form,

$$\xi_0 = \phi(x) + \frac{\pi^{\dot{\alpha}} \hat{\pi}^{\dot{\beta}}}{\langle \pi \hat{\pi} \rangle} \phi_{\dot{\alpha}\dot{\beta}}(x) + \frac{\pi^{\dot{\alpha}} \pi^{\dot{\beta}} \hat{\pi}^{\dot{\gamma}} \hat{\pi}^{\dot{\delta}}}{\langle \pi \hat{\pi} \rangle^2} \phi_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}(x) + \dots \tag{D.10}$$

the weight -1 form

$$\xi_\alpha = \frac{\hat{\pi}^{\dot{\alpha}}}{\langle \pi \hat{\pi} \rangle} \phi_{\alpha\dot{\alpha}}(x) + \frac{\pi^{\dot{\alpha}} \hat{\pi}^{\dot{\beta}} \hat{\pi}^{\dot{\gamma}}}{\langle \pi \hat{\pi} \rangle^2} \phi_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}(x) + \dots \tag{D.11}$$

and the weight -2 gauge function

$$f = \frac{\hat{\pi}^{\dot{\alpha}} \hat{\pi}^{\dot{\beta}}}{\langle \pi \hat{\pi} \rangle^2} f_{\dot{\alpha}\dot{\beta}}(x) + \dots \tag{D.12}$$

Note that an harmonic expansion for fields on the Riemann sphere is equal to the perhaps more familiar expansion in spherical harmonics since $SU(2) \sim SO(3)$. All coefficients in the expansions above are completely symmetric in the primed indices, since they would otherwise not be independent. In principle one has to expand the given action to all orders in the harmonic expansion. However, the gauge symmetry here comes to the rescue. Since

$$\bar{\partial}_0 = \langle \pi \hat{\pi} \rangle \pi^{\dot{\alpha}} \frac{\partial}{\partial \hat{\pi}^{\dot{\alpha}}} \tag{D.13}$$

one easily sees by explicit calculation that the gauge symmetry implies that the transformation

$$\xi_0 \rightarrow \xi_0 + \bar{\partial}_0 f^{-2} \tag{D.14}$$

can be used to set to zero all higher modes of the harmonic expansion (D.10). The field equations for ξ_α and $\bar{\xi}_\alpha$ in this gauge then set to zero all higher modes of these fields. The

action collapses therefore into just the space-time action written above. The harmonic expansion argument is of course equivalent to the gauge fixing argument given before. However, in addition it makes clear that one can invert the expansion by integrating both sides over the Riemann sphere. This leads to a lifting formula for the scalar fields,

$$\phi(x) = \int_{\mathbb{CP}^1} \xi_0 \quad \bar{\phi}(x) = \int_{\mathbb{CP}^1} \bar{\xi}_0 \tag{D.15}$$

By construction, this is invariant under the gauge transformations of the scalars which can be verified explicitly.

Penrose transform. Besides illustrating the action based derivation of the free field equations for a free massless scalar on twistor space, harmonic expansions also offer a convenient way to illustrate Penrose’s original observations about the map between cohomology classes and solutions to the wave equation in four dimensions directly. The field equation derived from (D.2) reads,

$$\bar{\partial}\xi = 0 \tag{D.16}$$

As noted before the solution to this equation up to gauge equivalence is precisely the definition of the Dolbeault cohomology class. In terms of the basis utilised throughout this article the equation can be written as (see (2.19))

$$\begin{aligned} \bar{\partial}_0 \xi_\alpha - \bar{\partial}_\alpha \xi_0 &= 0 \\ \bar{\partial}_\alpha \xi^\alpha &= 0 \end{aligned}$$

Inserting the harmonic expansions of (D.10) and (D.11) into these equations yields a system of equations which do not depend on π any more. The first few equations read,

$$\begin{aligned} \phi_{\alpha\dot{\alpha}}(p) - ip_{\alpha\dot{\alpha}}\phi(p) &= 0 \\ p_{\alpha\dot{\alpha}}\phi_{\dot{\beta}}^\alpha &= 0 \end{aligned} \tag{D.17}$$

for the first non-trivial one, and

$$\begin{aligned} \phi_{\alpha\dot{\alpha}\dot{\beta}\dot{\gamma}}(p) - ip_{\alpha\dot{\alpha}}\phi_{\dot{\beta}\dot{\gamma}}(p) &= 0 \\ p_{\alpha\dot{\alpha}}\phi_{\dot{\beta}\dot{\gamma}\dot{\delta}}^\alpha &= 0 \end{aligned} \tag{D.18}$$

for the second where the top line is symmetrised over the dotted indices. The higher order equations simply contain more symmetrised dotted indices. Above we have used a Fourier transform for only the space-time coordinate x . Therefore,

$$\bar{\partial}_\alpha = \pi^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} \rightarrow i\pi^{\dot{\alpha}} p_{\alpha\dot{\alpha}}. \tag{D.19}$$

By the same argument as above, the gauge symmetry shows that all the higher modes of the harmonic expansion of ξ_0 (D.10) are ‘pure gauge’. This starts with the $\phi_{\dot{\beta}\dot{\gamma}}$ in the equation directly above. Through the field equation, this translates immediately to almost all the harmonic modes of ξ_α . Above we used this gauge symmetry to set all these modes

to zero. In fact, the only non-trivial equation is (D.17) which is unaffected by the gauge transformations. Contracting the upper equation with $p^{\alpha\gamma}$ gives

$$p^2\phi(p) = 0 \tag{D.20}$$

which is of course the free wave equation of a scalar. So in the special case of a massless scalar field in four dimensions we have shown how the Penrose transform works.

D.1 Coupling the scalar twistor fields into Yang-Mills

We would now like to couple the scalars to the gauge field. The starting point is of course changing the transformation law for the scalars by adding a transformation term under the gauge symmetry

$$\xi^i \rightarrow \xi^i + f_j^{0,i}\xi^j \quad \bar{\xi}_j \rightarrow \bar{\xi}_j - \bar{\xi}_i f_j^{0,i} \tag{D.21}$$

Note that the only term allowed by a simple scaling argument is actually the weight zero gauge transformation parametrised by f^0 , not the weight -4 one. Of course, the action is now not invariant under this transformation.

$$\delta S_{\text{scalar}} = 2 \text{tr} \int d\Omega \wedge \bar{\xi}_i \wedge (\bar{\partial} f_j^{0,i}) \wedge \xi^j \tag{D.22}$$

It is an easy guess that one has to change the derivative in the action into a covariant one,

$$\delta_j^i \bar{\partial} \rightarrow \delta_j^i \bar{\partial} - i\sqrt{2}g B_j^i \tag{D.23}$$

This leads however to another problem: the action is now not invariant any more under the scalar gauge formations (D.3). This symmetry is crucial to keep the connection to space-time fields. A halfway solution to the problem is to turn the derivatives in that transformation into covariant ones by adding,

$$\delta \xi^i += -i\sqrt{2}g B_j^i f^{-2,j} \quad \delta \bar{\xi}_j += -i\sqrt{2}g \bar{f}_i^{-2} B_j^i \tag{D.24}$$

Although it is almost there now, the action is still not invariant under the gauge symmetry. It transforms however into a nice field strength,

$$\delta S_{\text{scalar}} = -2i\sqrt{2}g \text{tr} \int \xi_i \bar{f}_j^{-2} \wedge F^{ij} + \bar{\xi}_j f_i^{-2} \wedge F^{ij} \tag{D.25}$$

This suggests to modify the transformation law of the field \bar{B}

$$\delta \bar{B}_{ij} += -i\sqrt{2}g \xi_i \bar{f}_j^{-2} + -i\sqrt{2}g \bar{\xi}_j f_i^{-2} \tag{D.26}$$

so that the first term in the Yang-Mills action and the scalar action are invariant under all the gauge symmetries. The second ‘ C^2 ’ term of the Yang-Mills action is not any more because of the just added term to the transformation law of \bar{B} . Instead, this term transforms to

$$\delta S_{\text{YM}} = g \int_{\mathbb{CP}^1 \times \mathbb{CP}^1} (H \bar{B}_0 H^{-1})_1^{ij} \left(H (\xi_{0,i} \bar{f}_j^{-2} + \bar{\xi}_{0,j} f_i^{-2}) H^{-1} \right)_2 \langle \pi_1 \pi_2 \rangle^2 \tag{D.27}$$

This term can be cancelled by adding an interaction term to the action,

$$S += g \int_{(\mathbb{CP}^1)^3} (\bar{\xi}_0 H^{-1})_1 (H \bar{B}_0 H^{-1})_2 (H \xi_0)_3 \frac{\langle \pi_1 \pi_2 \rangle \langle \pi_2 \pi_3 \rangle}{\langle \pi_1 \pi_3 \rangle} \quad (\text{D.28})$$

The reason this works is that a partial integration of the variation will yield a delta function through

$$\bar{\partial}_0 \frac{1}{\langle \pi \kappa \rangle} = \delta(\langle \pi \kappa \rangle) \quad (\text{D.29})$$

for some spinor kappa. The added term again yields a problem through the transformation of \bar{B} , but applying the same remedy as above again yields,

$$S += \int_{(\mathbb{CP}^1)^4} (\bar{\xi}_0 H^{-1})_1 (H \xi_0)_2 (\bar{\xi}_0 H^{-1})_3 (H \xi_0)_4 \frac{\langle \pi_1 \pi_2 \rangle \langle \pi_3 \pi_4 \rangle}{\langle \pi_1 \pi_4 \rangle \langle \pi_2 \pi_3 \rangle} \quad (\text{D.30})$$

Now the action is invariant under the following transformations,

$$\xi \rightarrow \xi + \bar{\partial}_B f^{-2} \quad \bar{B} \rightarrow \bar{B} - i\sqrt{2}g[\bar{\xi}, f^{-2}] \quad (\text{D.31})$$

and

$$\bar{\xi} \rightarrow \bar{\xi} + \bar{\partial}_B \bar{f}^{-2} \quad \bar{B} \rightarrow \bar{B} - i\sqrt{2}g[\bar{\xi}, \bar{f}^{-2}] \quad (\text{D.32})$$

in addition to the gauge symmetry already present in the pure glue twistor action. However, since we started from a non-minimally coupled action (D.2) it has to be verified what the coupled action has the right space-time interpretation. It is easy to check that gauge-fixing space-time gauge in the action will yield a ϕ^4 interaction term in the space-time Lagrangian. The reason the Noether procedure is ambiguous in this case is that there exists a separate term invariant under the requested symmetries,

$$S_{\phi^4} = g^2 \int_{(\mathbb{CP}^1)^4} (\bar{\xi}_0 H^{-1})_1 (H \xi_0)_2 (\bar{\xi}_0 H^{-1})_3 (H \xi_0)_4. \quad (\text{D.33})$$

Hence this term can be added to the action with a constant designed to eliminate the ϕ^4 term in space-time gauge. Of course, this is the procedure employed in the main text.

D.2 Lifting formula

Note that (D.15) is now not invariant any more under the gauge transformation. This can easily be remedied by adding frames,

$$\phi(x) = \int_{\mathbb{CP}^1} H^{-1} \xi_0 \quad \bar{\phi}(x) = \int_{\mathbb{CP}^1} \bar{\xi}_0 H \quad (\text{D.34})$$

In effect, the frames are needed to gauge-transform back to the case where B_0 is zero. These are the formulae used in the main text (3.41). Lifting a ϕ^4 term directly using these gives (D.33).

From this appendix it should be clear that in principle scalars in any representation of the gauge group can be lifted to twistor space. In particular, it can be verified (or easily guessed!) that in the adjoint representation the lifting formulae are

$$\phi(x) = \int_{\mathbb{CP}^1} H^{-1} \xi_0 H \quad \bar{\phi}(x) = \int_{\mathbb{CP}^1} H^{-1} \bar{\xi}_0 H \quad (\text{D.35})$$

In more general cases, one needs the Wilson link operator in the required representation.

From the starting point (D.2) one can see that massless fields of in principle *any* spin can be coupled to Yang-Mills theory through a similar Noether procedure. In practice this needs a bit of care when one needs to construct interaction terms to balance the transformation of \bar{B} for higher spin fields. In principle this procedure yields consistent Lagrangians for higher spin fields coupled to Yang-Mills theory. Following the same gauge fixing steps as before, it also easily yields MHV amplitudes for these. A more pedestrian application of these observations would be for spin $\frac{1}{2}$ fermions. We hope to return to this issue in future work.

References

- [1] S.J. Parke and T.R. Taylor, *An amplitude for n gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459.
- [2] F.A. Berends and W.T. Giele, *Recursive calculations for processes with n gluons*, *Nucl. Phys.* **B 306** (1988) 759.
- [3] V.P. Nair, *A current algebra for some gauge theory amplitudes*, *Phys. Lett.* **B 214** (1988) 215.
- [4] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [[hep-th/0312171](#)].
- [5] F. Cachazo, P. Svrček and E. Witten, *MHV vertices and tree amplitudes in gauge theory*, *JHEP* **09** (2004) 006 [[hep-th/0403047](#)].
- [6] G. Georgiou and V.V. Khoze, *Tree amplitudes in gauge theory as scalar MHV diagrams*, *JHEP* **05** (2004) 070 [[hep-th/0404072](#)].
- [7] G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *JHEP* **07** (2004) 048 [[hep-th/0407027](#)].
- [8] J.-B. Wu and C.-J. Zhu, *MHV vertices and scattering amplitudes in gauge theory*, *JHEP* **07** (2004) 032 [[hep-th/0406085](#)].
- [9] L.J. Dixon, E.W.N. Glover and V.V. Khoze, *MHV rules for Higgs plus multi-gluon amplitudes*, *JHEP* **12** (2004) 015 [[hep-th/0411092](#)].
- [10] S.D. Badger, E.W.N. Glover and V.V. Khoze, *MHV rules for Higgs plus multi-parton amplitudes*, *JHEP* **03** (2005) 023 [[hep-th/0412275](#)].
- [11] Z. Bern, D. Forde, D.A. Kosower and P. Mastrolia, *Twistor-inspired construction of electroweak vector boson currents*, *Phys. Rev.* **D 72** (2005) 025006 [[hep-ph/0412167](#)].
- [12] I. Bena, Z. Bern and D.A. Kosower, *Twistor-space recursive formulation of gauge theory amplitudes*, *Phys. Rev.* **D 71** (2005) 045008 [[hep-th/0406133](#)].
- [13] D.A. Kosower, *Next-to-maximal helicity violating amplitudes in gauge theory*, *Phys. Rev.* **D 71** (2005) 045007 [[hep-th/0406175](#)].
- [14] K.J. Ozeren and W.J. Stirling, *MHV techniques for QED processes*, *JHEP* **11** (2005) 016 [[hep-th/0509063](#)].
- [15] M. Dinsdale, M. Ternick and S. Weinzierl, *A comparison of efficient methods for the computation of Born gluon amplitudes*, *JHEP* **03** (2006) 056 [[hep-ph/0602204](#)].

- [16] A. Brandhuber, B.J. Spence and G. Travaglini, *One-loop gauge theory amplitudes in $N = 4$ super Yang-Mills from MHV vertices*, *Nucl. Phys. B* **706** (2005) 150 [[hep-th/0407214](#)].
- [17] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *A twistor approach to one-loop amplitudes in $N = 1$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **706** (2005) 100 [[hep-th/0410280](#)].
- [18] C. Quigley and M. Rozali, *One-loop MHV amplitudes in supersymmetric gauge theories*, *JHEP* **01** (2005) 053 [[hep-th/0410278](#)].
- [19] J. Bedford, A. Brandhuber, B.J. Spence and G. Travaglini, *Non-supersymmetric loop amplitudes and MHV vertices*, *Nucl. Phys. B* **712** (2005) 59 [[hep-th/0412108](#)].
- [20] A. Brandhuber, B. Spence and G. Travaglini, *From trees to loops and back*, *JHEP* **01** (2006) 142 [[hep-th/0510253](#)].
- [21] S.D. Badger, E.W.N. Glover and K. Risager, *One-loop phi-MHV amplitudes using the unitarity bootstrap*, *JHEP* **07** (2007) 066 [[arXiv:0704.3914](#)].
- [22] E.W.N. Glover, P. Mastrolia and C. Williams, *One-loop phi-MHV amplitudes using the unitarity bootstrap: the general helicity case*, [arXiv:0804.4149](#).
- [23] R. Britto, F. Cachazo and B. Feng, *New recursion relations for tree amplitudes of gluons*, *Nucl. Phys. B* **715** (2005) 499 [[hep-th/0412308](#)].
- [24] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [[hep-th/0501052](#)].
- [25] K. Risager, *A direct proof of the CSW rules*, *JHEP* **12** (2005) 003 [[hep-th/0508206](#)].
- [26] A. Gorsky and A. Rosly, *From Yang-Mills Lagrangian to MHV diagrams*, *JHEP* **01** (2006) 101 [[hep-th/0510111](#)].
- [27] P. Mansfield, *The Lagrangian origin of MHV rules*, *JHEP* **03** (2006) 037 [[hep-th/0511264](#)].
- [28] R. Boels, L. Mason and D. Skinner, *From twistor actions to MHV diagrams*, *Phys. Lett. B* **648** (2007) 90 [[hep-th/0702035](#)].
- [29] A. Brandhuber, B. Spence and G. Travaglini, *Amplitudes in pure Yang-Mills and MHV diagrams*, *JHEP* **02** (2007) 088 [[hep-th/0612007](#)].
- [30] R. Boels, *A quantization of twistor Yang-Mills theory through the background field method*, *Phys. Rev. D* **76** (2007) 105027 [[hep-th/0703080](#)].
- [31] A. Brandhuber, B. Spence, G. Travaglini and K. Zoubos, *One-loop MHV Rules and Pure Yang-Mills*, *JHEP* **07** (2007) 002 [[arXiv:0704.0245](#)].
- [32] J.H. Eittle, C.-H. Fu, J.P. Fudger, P.R.W. Mansfield and T.R. Morris, *S-matrix equivalence theorem evasion and dimensional regularisation with the canonical MHV Lagrangian*, *JHEP* **05** (2007) 011 [[hep-th/0703286](#)].
- [33] R. Boels, L. Mason and D. Skinner, *Supersymmetric gauge theories in twistor space*, *JHEP* **02** (2007) 014 [[hep-th/0604040](#)].
- [34] H. Feng and Y.-t. Huang, *MHV lagrangian for $N = 4$ super Yang-Mills*, [hep-th/0611164](#).
- [35] S. Ananth and S. Theisen, *KLT relations from the Einstein-Hilbert Lagrangian*, *Phys. Lett. B* **652** (2007) 128 [[arXiv:0706.1778](#)].

- [36] L.J. Mason and M. Wolf, *A twistor action for $N = 8$ self-dual supergravity*, arXiv:0706.1941.
- [37] R. Boels and C. Schwinn, *CSW rules for a massive scalar*, *Phys. Lett. B* **662** (2008) 80 [arXiv:0712.3409].
- [38] C. Schwinn and S. Weinzierl, *SUSY Ward identities for multi-gluon helicity amplitudes with massive quarks*, *JHEP* **03** (2006) 030 [hep-th/0602012].
- [39] L.J. Dixon, *Calculating scattering amplitudes efficiently*, hep-ph/9601359.
- [40] S.D. Badger, E.W.N. Glover, V.V. Khoze and P. Svrček, *Recursion relations for gauge theory amplitudes with massive particles*, *JHEP* **07** (2005) 025 [hep-th/0504159].
- [41] D. Forde and D.A. Kosower, *All-multiplicity amplitudes with massive scalars*, *Phys. Rev. D* **73** (2006) 065007 [hep-th/0507292].
- [42] P. Ferrario, G. Rodrigo and P. Talavera, *Compact multigluonic scattering amplitudes with heavy scalars and fermions*, *Phys. Rev. Lett.* **96** (2006) 182001 [hep-th/0602043].
- [43] C. Schwinn and S. Weinzierl, *On-shell recursion relations for all Born QCD amplitudes*, *JHEP* **04** (2007) 072 [hep-ph/0703021].
- [44] J.H. Eittle and T.R. Morris, *Structure of the MHV-rules Lagrangian*, *JHEP* **08** (2006) 003 [hep-th/0605121].
- [45] L.J. Mason, *Twistor actions for non-self-dual fields: a derivation of twistor-string theory*, *JHEP* **10** (2005) 009 [hep-th/0507269].
- [46] G. Chalmers and W. Siegel, *Simplifying algebra in Feynman graphs. II: spinor helicity from the spacecone*, *Phys. Rev. D* **59** (1999) 045013 [hep-ph/9801220].
- [47] N.M.J. Woodhouse, *Real methods in twistor theory*, *Class. and Quant. Grav.* **2** (1985) 257.
- [48] G. Chalmers and W. Siegel, *The self-dual sector of QCD amplitudes*, *Phys. Rev. D* **54** (1996) 7628 [hep-th/9606061].
- [49] G.'t Hooft and M.J.G. Veltman, *Diagrammar*, *NATO Adv. Study Inst. Ser. B Phys.* **4** (1974) 177.
- [50] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *Harmonic superspace*, Cambridge University Press, Cambridge U.K. (2001).
- [51] G. Chalmers and W. Siegel, *Simplifying algebra in Feynman graphs. I: spinors*, *Phys. Rev. D* **59** (1999) 045012 [hep-ph/9708251].
- [52] E. Sokatchev, *An action for $N = 4$ supersymmetric self-dual Yang-Mills theory*, *Phys. Rev. D* **53** (1996) 2062 [hep-th/9509099].
- [53] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One-loop self-dual and $N = 4$ super Yang-Mills*, *Phys. Lett. B* **394** (1997) 105 [hep-th/9611127].
- [54] J.H. Eittle, T.R. Morris and Z. Xiao, *The MHV QCD Lagrangian*, arXiv:0805.0239.
- [55] NLO MULTILEG WORKING GROUP collaboration, Z. Bern et al., *The NLO multileg working group: summary report*, arXiv:0803.0494.
- [56] Z. Bern, L.J. Dixon and D.A. Kosower, *Progress in one-loop QCD computations*, *Ann. Rev. Nucl. Part. Sci.* **46** (1996) 109 [hep-ph/9602280].

- [57] R.P. Feynman, *Closed loop and tree diagrams*, in *Magic Without Magic*, J.R. Klauder ed., San Francisco (1972) page 355.
- [58] W.A. Bardeen, *Selfdual Yang-Mills theory, integrability and multiparton amplitudes*, *Prog. Theor. Phys. Suppl.* **123** (1996) 1.
- [59] G. Leibbrandt, *Introduction to Noncovariant Gauges*, *Rev. Mod. Phys.* **59** (1987) 1067.
- [60] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, *D-dimensional unitarity cut method*, *Phys. Lett.* **B 645** (2007) 213 [[hep-ph/0609191](#)].
- [61] I. Jack, D.R.T. Jones and K.L. Roberts, *Equivalence of dimensional reduction and dimensional regularization*, *Z. Physik C* **63** (1994) 151 [[hep-ph/9401349](#)].
- [62] F. Wilczek, *Decays of heavy vector mesons into Higgs particles*, *Phys. Rev. Lett.* **39** (1977) 1304.
- [63] M.A. Shifman, A.I. Vainshtein, M.B. Voloshin and V.I. Zakharov, *Low-energy theorems for Higgs boson couplings to photons*, *Sov. J. Nucl. Phys.* **30** (1979) 711 [*Yad. Fiz.* **30** (1979) 1368].
- [64] C.F. Berger, V. Del Duca and L.J. Dixon, *Recursive construction of Higgs+multiparton loop amplitudes: the last of the phi-nite loop amplitudes*, *Phys. Rev.* **D 74** (2006) 094021 [*Erratum ibid.* **D 76** (2007) 099901] [[hep-ph/0608180](#)].